Electromagnetic properties of nuclei: from few- to many-body systems

Lecture3

Multipole Expansion

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Multipole decomposition

Since the intrinsic states of the nucleus can be classified according to the total angular momentum, it is very useful to perform a multipole decomposition of the charge and of the current operators, where each multipole transfers a definite angular momentum J.

The advantage of this approach is that for example if you are representing a wave function on a spherical basis, then you can only construct those states that have the quantum number J as connected by your multipole operator.

Reduces complexity of each nuclear matrix element

Charge operator

$$\rho(\mathbf{x}) = e\sum_i \frac{1+\tau_i^3}{2}\,\delta(\mathbf{x}-\mathbf{r}_i)$$

⚠ These are intended as operators now

$$\Big\downarrow \text{FF}$$

$$\rho(\mathbf{q}) = \int d^3x\; e^{i\mathbf{q}\cdot\mathbf{x}}\;\rho(\mathbf{x})$$

operator that enters in the nuclear ME

$$\rho(\mathbf{q}) = e\sum_i \frac{1+\tau_i^3}{2}\int d^3x\; e^{i\mathbf{q}\cdot\mathbf{x}}\;\delta(\mathbf{x}-\mathbf{r}_i)$$

$$= e\sum_i \frac{1+\tau_i^3}{2}\; e^{i\mathbf{q}\cdot\mathbf{r}_i}$$

One body operator

Charge operator

Spatial part, single coordinate   omitting i-index

$e^{i\mathbf{q}\cdot\mathbf{r}}$ ⟶ scalar function, that depends on (r,θ,ϕ)

Any function that depends on angles can be expanded in spherical harmonics, as they are a complete set of basis states

$$f(\theta,\phi)=\sum_{J\mu}a_{J\mu}Y^{J*}_{\mu}(\theta,\phi)$$

with

$$a_{J\mu}=\int d\theta\int d\phi\; f(\theta,\phi)Y^{J}_{\mu}\;(\theta,\phi)$$

Charge operator

Spatial part, single coordinate omitting i-index

$e^{i\mathbf{q}\cdot\mathbf{r}}$ ⟶ scalar function, that depends on (r,θ,ϕ)

Any function that depends on angles can be expanded in spherical harmonics, as they are a complete set of basis states

$$f(r,\theta,\phi)=\sum_{J\mu} a_{J\mu}(r) Y_{\mu}^{J^{*}}(\theta,\phi)$$

See Varshalovich book

with

$$a_{J\mu}(r)=\int d\theta \int d\phi \; f(r,\theta,\phi) Y_{\mu}^{J}\;(\theta,\phi)$$

Charge operator

Spatial part, single coordinate   omitting i-index

$e^{i\mathbf{q}\cdot\mathbf{r}}$ ⟶ scalar function, that depends on (r,θ,ϕ)

Any function that depends on angles can be expanded in spherical harmonics, as they are a complete set of basis states

Plane wave expansion in spherical harmonics

$$e^{i\mathbf{q}\cdot\mathbf{r}} = \boxed{4\pi}\sum_{J\mu}\boxed{i^J j_J(qr)}\,Y_\mu^{J^*}(\hat{q})\boxed{Y_\mu^J(\hat{r})}$$

☐ expansion coefficient

Radial part

Angular part

Operator that carries angular momentum J

$\Rightarrow$ $j_J(qr) \quad Y_\mu^J(\hat{r})$

Charge operator

Spatial part, single coordinate   omitting i-index

$e^{i\mathbf{q}\cdot\mathbf{r}} \longrightarrow$ scalar function, that depends on (r,θ,ϕ)

Any function that depends on angles can be expanded in spherical harmonics, as they are a complete set of basis states

Plane wave expansion in spherical harmonics

$$e^{i\mathbf{q}\cdot\mathbf{r}} = 4\pi \sum_{J\mu} i^J \underset{\uparrow}{j_J(qr)} Y_\mu^{J^*}(\hat{q}) \underset{\uparrow}{Y_\mu^J(\hat{r})}$$

Radial part   Angular part

Operator that carries angular momentum J

$$\Rightarrow \quad C_{J\mu} \propto \sum_i \; j_J(qr_i) \;\; Y_\mu^J(\hat{r}_i)$$

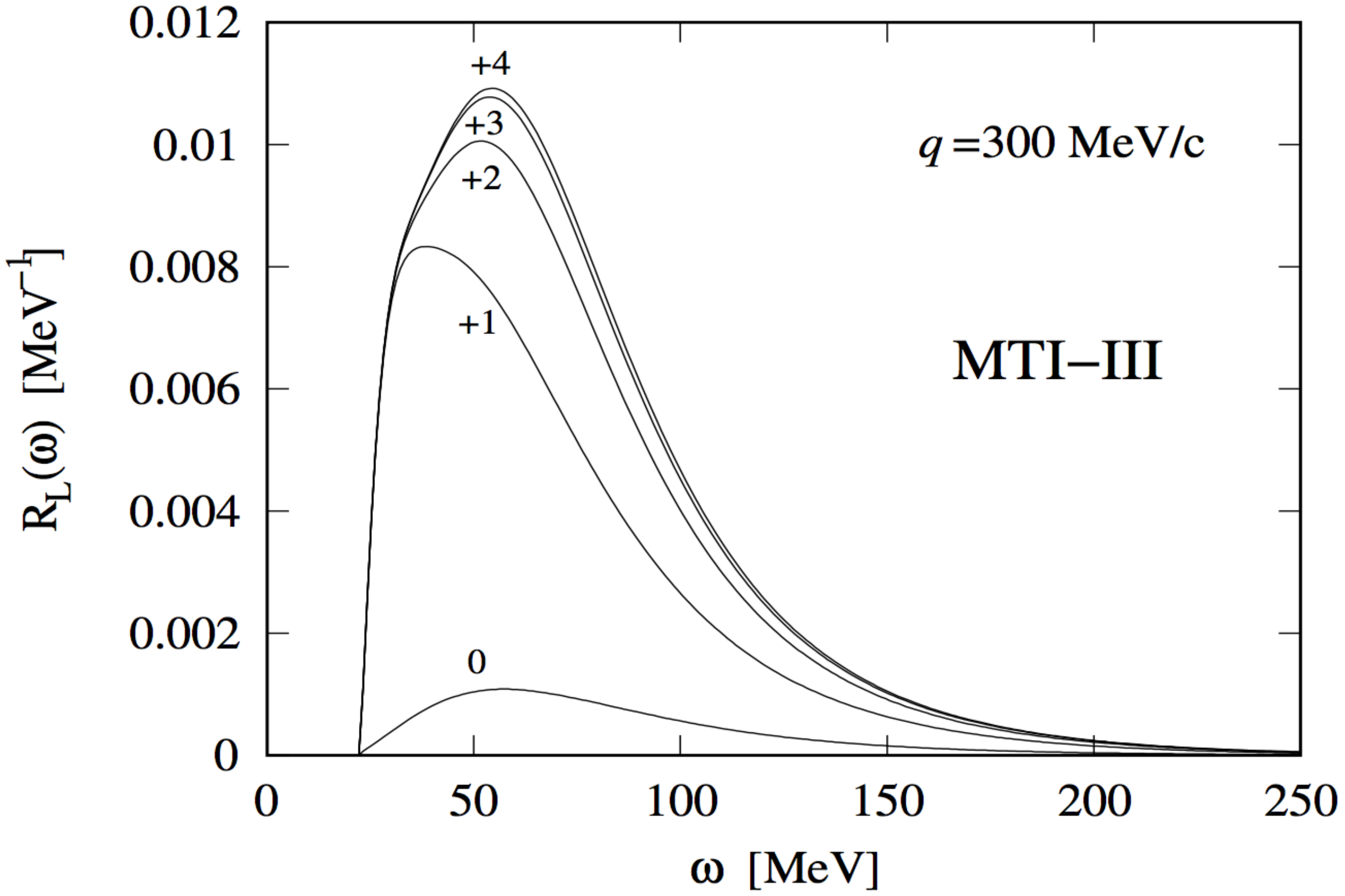
Coulomb multipole

For A-nucleons

Practical Example ^{4}He

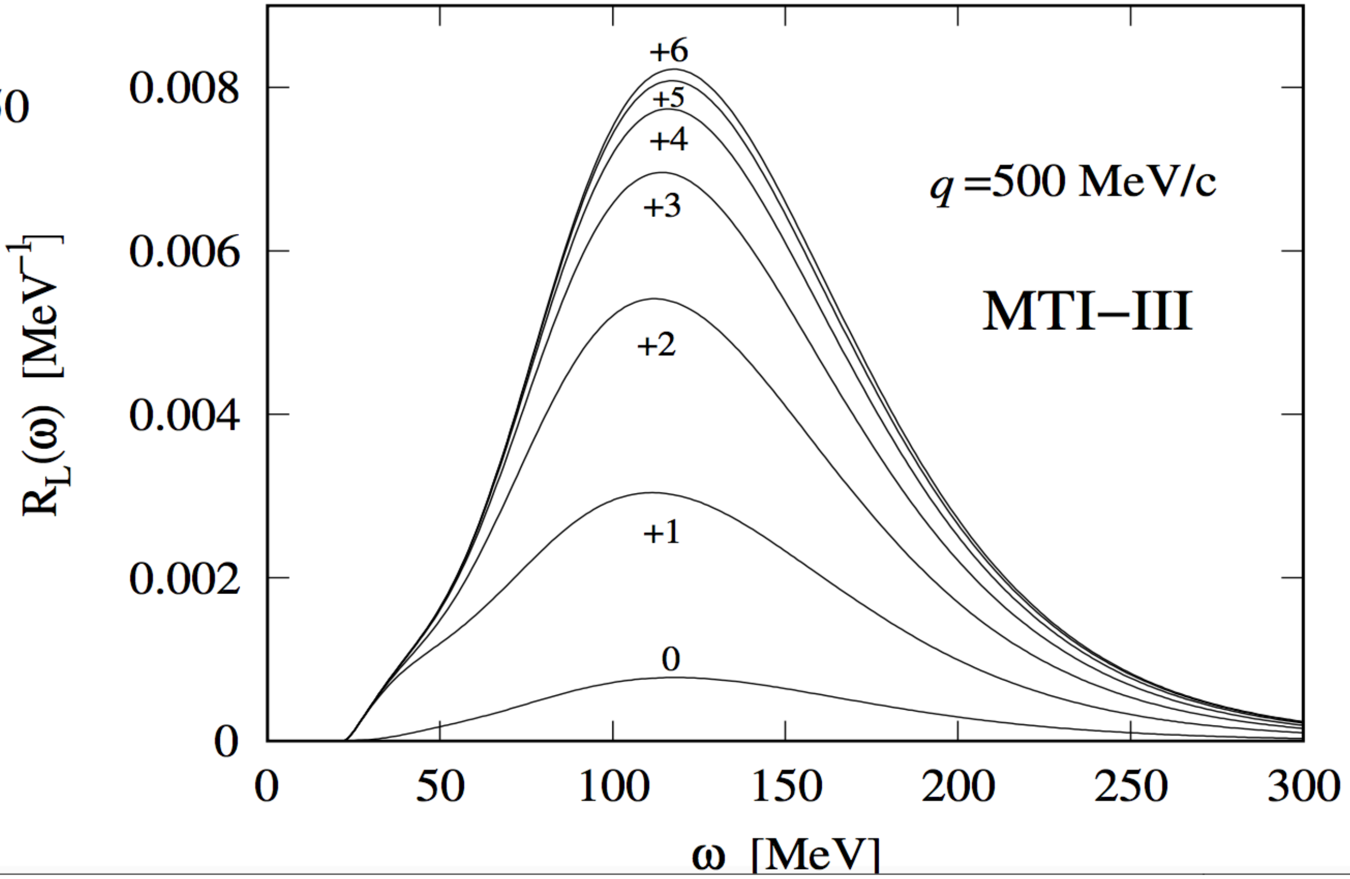
Recursive sum of Coulomb multipoles

S.B. *et al.*, PRC **76**, 014003 (2007)



At low q the Coulomb dipole of oder 1 dominates

The higher the momentum transfer, the slower the multipole convergence

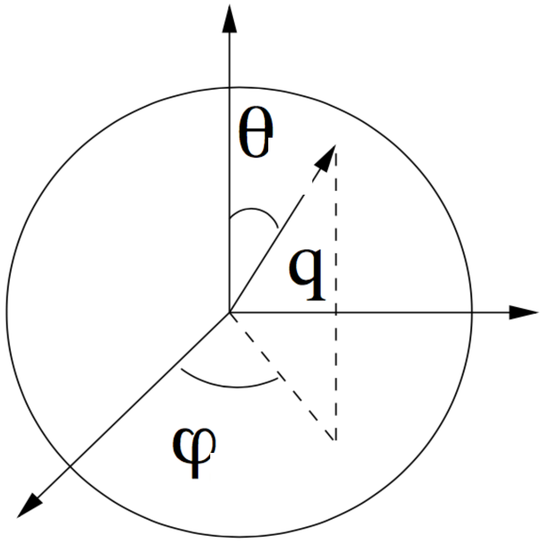


Current operator

Since the current operator is a vector, the expansion is done in terms of the vector spherical harmonics

$$\mathbf{Y}^{\mu}_{Jl1}(\hat{q}) = \sum_{m\xi} \langle l1J|m\xi\mu\rangle\, Y^{l}_{m}(\hat{q})\mathbf{e}_{\xi}$$

Unit vector in the spherical basis



$$\mathbf{e}_1 = -\frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y)$$

$$\mathbf{e}_0 = \mathbf{e}_z$$

$$\mathbf{e}_{-1} = \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y)$$

The vector spherical harmonics form a complete set on the unit sphere

$$\int d\hat{q}'\; \mathbf{Y}^{\mu' *}_{J'l'1}(\hat{q}') \cdot \mathbf{Y}^{\mu}_{Jl1}(\hat{q}') = \delta_{JJ'}\delta_{ll'}\delta_{\mu,\mu'}$$

Current operator

Multipole expansion of the current operator

$$\mathbf{J}\left(\mathbf{q}\right) = 4\pi \sum_{lJ\mu} J^{\mu}_{Jl}(q)\mathbf{Y}^{\mu *}_{Jl1}(\hat{q})$$

with $J^{\mu}_{Jl}(q) = \frac{1}{4\pi}\int d\hat{q}\,'\mathbf{J}\left(\mathbf{q}'\right)\cdot\mathbf{Y}^{\mu}_{Jl1}(\hat{q}\,')$

According to angular momentum rules l=J-1,J,J+1 ⇒ separate according to parity

$$\mathbf{J}\left(\mathbf{q}\right) = \sum_{J\mu}\left(\mathbf{J}^{el}_{J\mu}(\mathbf{q}) + \mathbf{J}^{mag}_{J\mu}(\mathbf{q})\right)$$

$$\mathbf{J}^{el}_{J\mu}(\mathbf{q}) = 4\pi\left(J^{\mu}_{JJ-1}(q)\mathbf{Y}^{\mu *}_{JJ-11}(\hat{q}) + J^{\mu}_{JJ+1}(q)\mathbf{Y}^{\mu *}_{JJ+11}(\hat{q})\right)$$

Electric multipoles

parity $(-1)^J$

$$\mathbf{J}^{mag}_{J\mu}(\mathbf{q}) = 4\pi J^{\mu}_{JJ}(q)\mathbf{Y}^{\mu *}_{JJ1}(\hat{q})$$

Magnetic multipoles parity $(-1)^{J+1}$

Current operator

The expression for the electric multipole can be rewritten as

$$\mathbf{J}^{el}_{J\mu}(\mathbf{q}) = \hat{\mathbf{q}} Y^{J*}_{\mu}(\hat{q}) \int d\hat{q}' \left(\hat{\mathbf{q}}' \cdot \mathbf{J}\left(\mathbf{q}'\right)\right) Y^{J}_{\mu}(\hat{q}') + \left(\hat{\mathbf{q}} \times \mathbf{Y}^{\mu *}_{JJ1}(\hat{q})\right) \int d\hat{q}' \left(\hat{\mathbf{q}}' \times \mathbf{Y}^{\mu}_{JJ1}(\hat{q}')\right) \mathbf{J}\left(\mathbf{q}'\right)$$

Longitudinal part of the current

Transverse part of the current

Introducing longitudinal and transverse electric multipoles and magnetic multipoles (transverse only due to $\hat{\mathbf{q}} \cdot \mathbf{Y}^{\mu}_{JJ1}(\hat{q}) = 0$)

$$L^{el}_{J\mu}(q) = \frac{1}{4\pi} \int d\hat{q}' \left(\hat{\mathbf{q}}' \cdot \mathbf{J}\left(\mathbf{q}'\right)\right) Y^{J}_{\mu}(\hat{q}')$$

$$T^{el}_{J\mu}(q) = \frac{i}{4\pi} \int d\hat{q}' \left(\hat{\mathbf{q}}' \times \mathbf{Y}^{\mu}_{JJ1}(\hat{q}')\right) \cdot \mathbf{J}\left(\mathbf{q}'\right)$$

$$T^{mag}_{J\mu}(q) = \frac{1}{4\pi} \int d\hat{q}' \mathbf{J}\left(\mathbf{q}'\right) \cdot \mathbf{Y}^{\mu}_{JJ1}(\hat{q}')$$

NB: for every piece of em current (convection, spin, MEC) one can calculate these multipoles

Current operator

Choosing the z-axis as the direction of propagation of the photon momentum

$\mathbf{q} = q\mathbf{e}_z = q\mathbf{e}_0$ then

$$\mathbf{Y}^{\mu}_{Jl1} = \langle l1J|0\mu\mu\rangle \frac{\hat{l}}{\sqrt{4\pi}}\mathbf{e}_\mu$$

Substitute all of these in the expression of the current in terms of longitudinal, electric and magnetic multipoles

$$\mathbf{J}(\mathbf{q}) = \sum_{J\mu}\sqrt{4\pi}\hat{J}\left[L^{el}_{J\mu}(q)\mathbf{e}_0 + \mu\langle J1J|0\mu\mu\rangle T^{el}_{J\mu}(q)\mathbf{e}^*_\mu\right] + \sum_{J\mu}\sqrt{4\pi}\hat{J}\langle J1J|0\mu\mu\rangle T^{mag}_{J\mu}(q)\mathbf{e}^*_\mu$$

As in the nuclear matrix elements what we need is $\mathbf{e}_\lambda \cdot \mathbf{J}(\mathbf{q})$ then we rewrite as

$$\boxed{\mathbf{e}_\lambda \cdot \mathbf{J}(\mathbf{q}) = (-)^\lambda\sqrt{2\pi(1+\delta_{\lambda 0})}\sum_J \hat{J}\left[L^{el}_{J\lambda}(q)\delta_{\lambda 0} + \left(T^{el}_{J\lambda}(q) + \lambda T^{mag}_{J\lambda}(q)\right)\delta_{|\lambda|1}\right]}$$

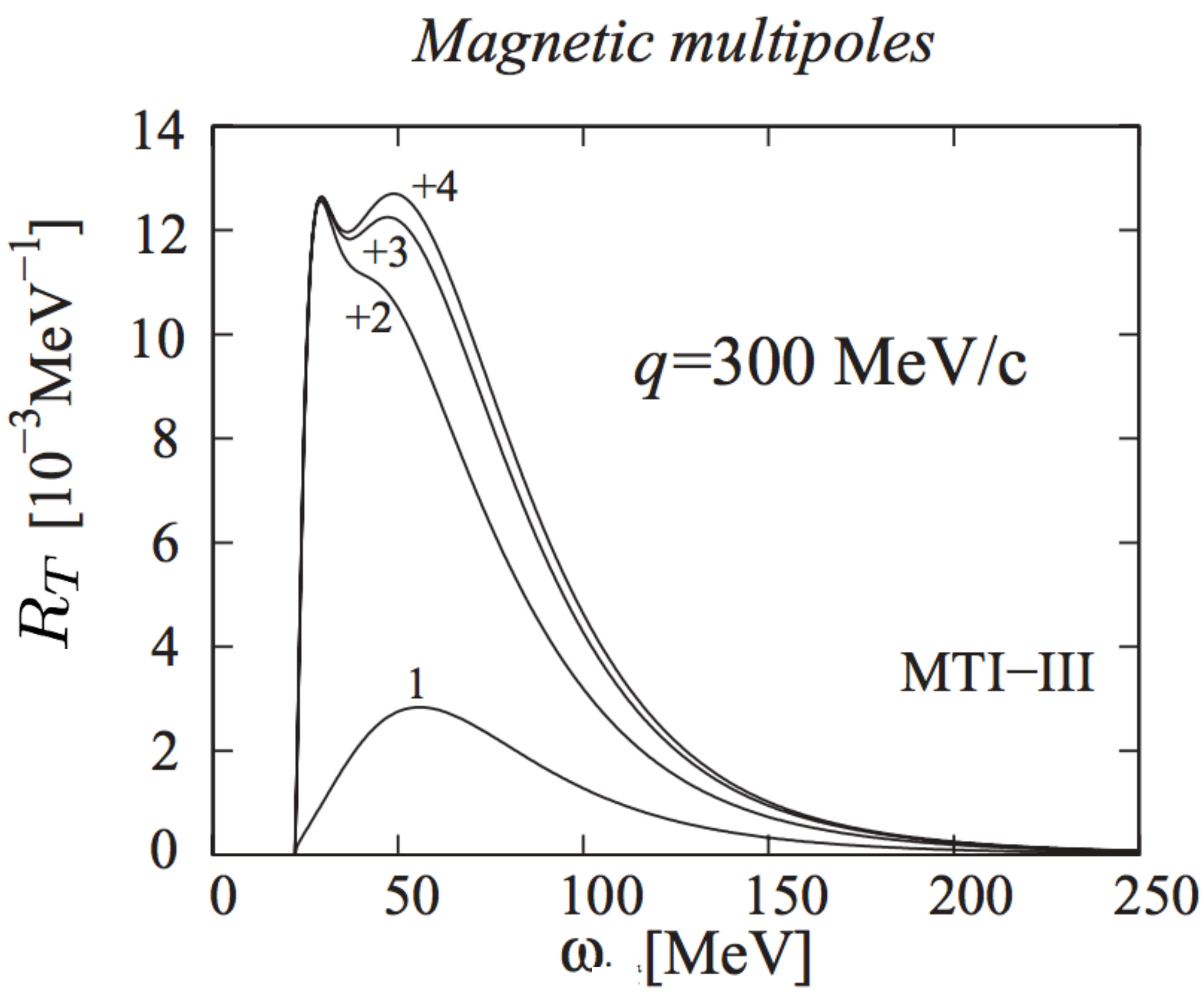
Multipole decomposition of the current operator

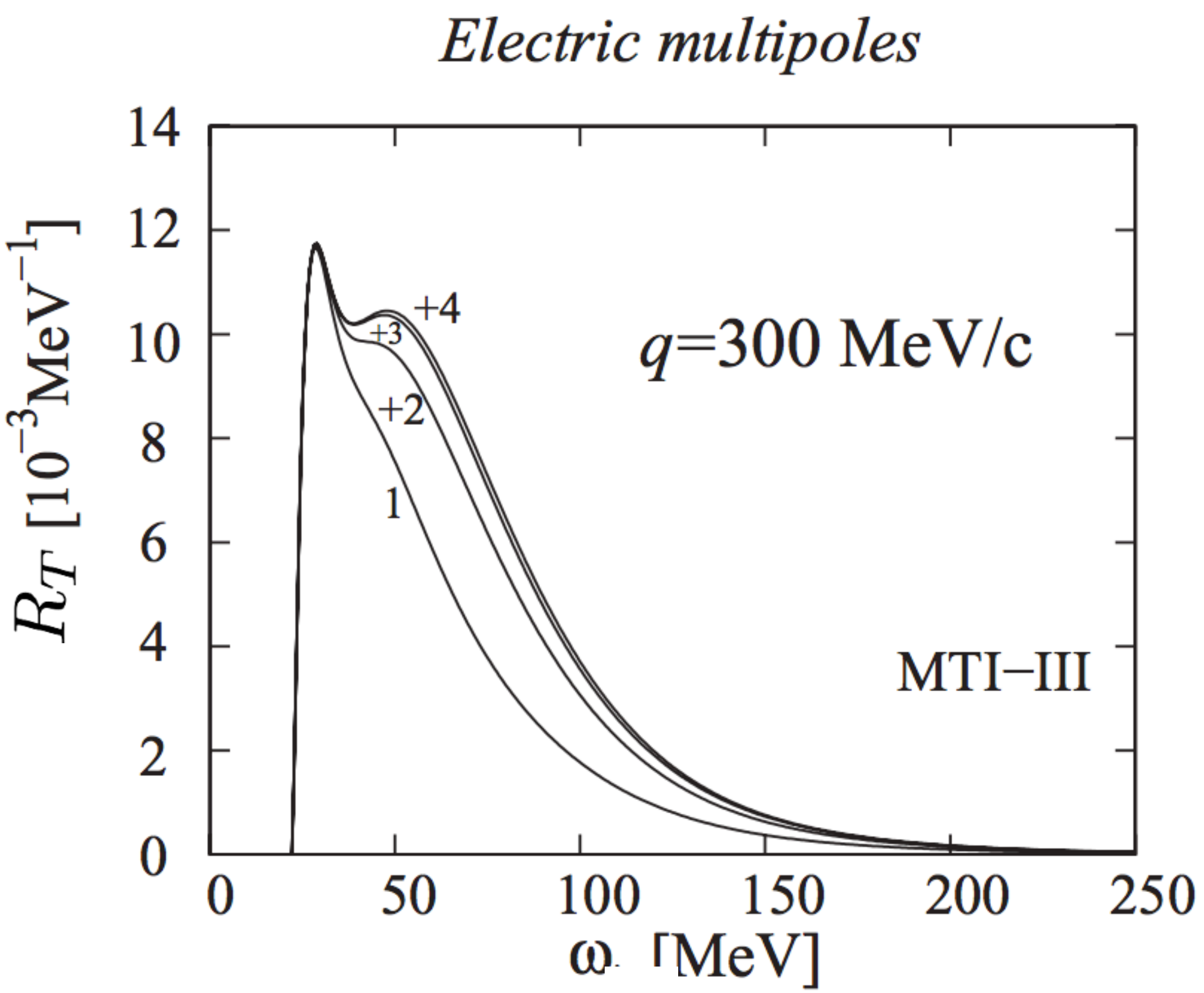
Practical Example ^{4}He

$$\mathbf{e}_\lambda \cdot \mathbf{J}\left(\mathbf{q}\right) = (-)^\lambda \sqrt{2\pi(1+\delta_{\lambda 0})} \sum_J \hat{J} \, [L^{el}_{J\lambda}(q)\delta_{\lambda 0} + \boxed{\left(T^{el}_{J\lambda}(q) + \lambda T^{mag}_{J\lambda}(q)\right) \delta_{|\lambda|1}]}$$

Recursive sum of transverse multipoles

S.B. *et al.*, PRC **76**, 014003 (2007)





The Siegert theorem

Let us look again at the form of the transverse electric multipoles

$$T^{el}_{J\mu}(q) = \frac{i}{4\pi}\int d\hat{q}'\,(\hat{\mathbf{q}}'\times\mathbf{Y}^{\mu}_{JJ1}(\hat{q}'))\cdot\mathbf{J}\,(\mathbf{q}')$$

using the property: $\hat{\mathbf{q}}\times\mathbf{Y}^{\mu}_{JJ1}(\hat{q}) = i\sqrt{\frac{J+1}{J}}\hat{\mathbf{q}}Y^{J}_{\mu}(\hat{q}) + i\frac{\hat{J}}{\sqrt{J}}\mathbf{Y}^{\mu}_{JJ+11}(\hat{q})$

$$T^{el}_{J\mu}(q) = \boxed{-\frac{1}{4\pi}\int d\hat{q}'\,\Big[\sqrt{\frac{J+1}{J}}\hat{\mathbf{q}}'\cdot\mathbf{J}(\mathbf{q}')\,Y^{J}_{\mu}(\hat{q}')} + \frac{\hat{J}}{\sqrt{J}}\mathbf{Y}^{\mu}_{JJ+11}(\hat{q}')\cdot\mathbf{J}(\mathbf{q}')\Big]$$

Siegert operator

Can be related to a Coulomb multipole via the use of the continuity equation

Correction to the Siegert operator

$$\mathbf{q}\cdot\mathbf{J}(\mathbf{q}) = \omega\rho(\mathbf{q})$$

Siegert theorem

$$\boxed{T^{el}_{J\mu}(q) = -\frac{1}{4\pi}\sqrt{\frac{J+1}{J}}\frac{\omega}{q}\int d\hat{q}'\rho(\mathbf{q}')Y^{J}_{\mu}(\hat{q}')} - \frac{1}{4\pi}\frac{\hat{J}}{\sqrt{J}}\int d\hat{q}'\cancel{\mathbf{Y}^{\mu}_{JJ+11}}(\hat{q}')\cdot\mathbf{J}(\mathbf{q}')$$

$C_{J\mu}$

Negligible at low q

Practical Example J=1 ^{4}He

$$T^{el}_{J\mu}(q) = \boxed{-\frac{1}{4\pi}\int d\hat{q}'\left[\sqrt{\frac{J+1}{J}}\,\hat{\mathbf{q}}'\cdot\mathbf{J}(\mathbf{q}')\;Y^J_\mu(\hat{q}')\right.}$$

Siegert operator	———	via continuity equation, relate to $C_{1\mu}$
Convection current	········	$\hat{\mathbf{q}}'\cdot\mathbf{J}_c(\mathbf{q}')$
MEC	–·–·–	$\hat{\mathbf{q}}'\cdot\mathbf{J}_{\mathrm{MEC}}(\mathbf{q}')$
MEC+convection	- - - - -	$\hat{\mathbf{q}}'\cdot(\mathbf{J}_c(\mathbf{q}')+\mathbf{J}_{\mathrm{MEC}}(\mathbf{q}'))$

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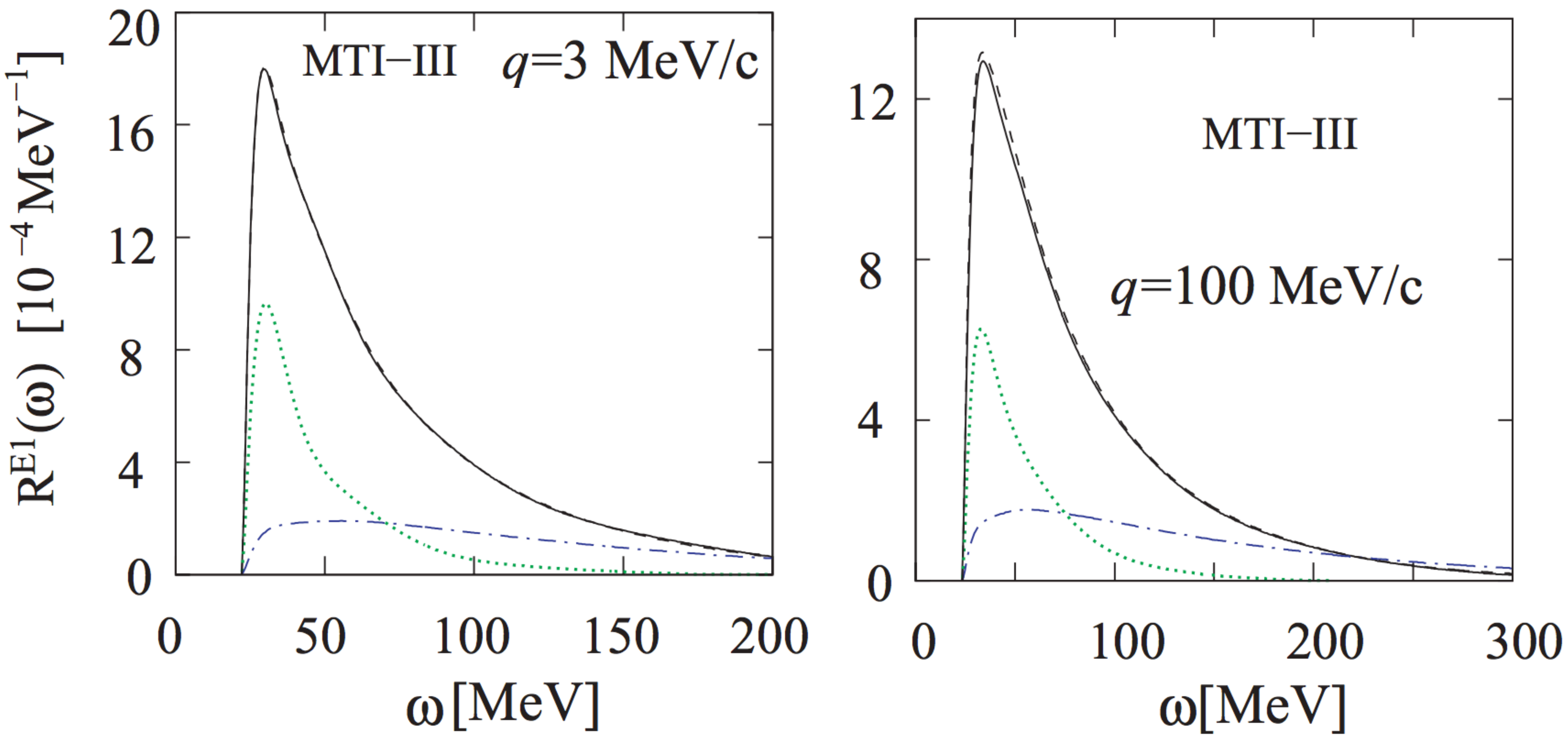
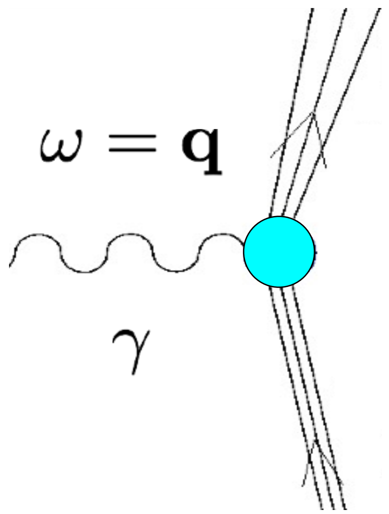


Photoabsorption

Recap:



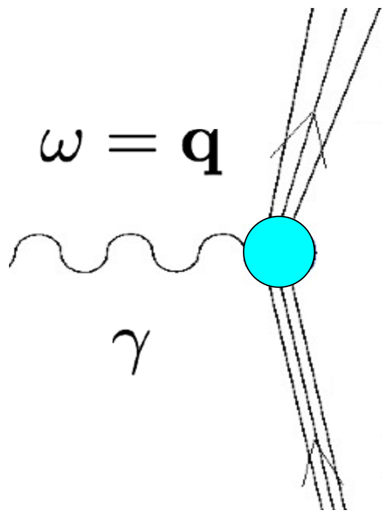
$$\sigma = \sigma_M \left[\cancel{\frac{Q^4}{\mathbf{q}^4} R_L(\omega, \mathbf{q})} + \left(\frac{Q^2}{2\mathbf{q}^2} + \tan^2 \frac{\theta}{2} \right) \overset{\omega = \mathbf{q}}{R_T(\omega, \mathbf{q})} \right]$$

~~Longitudinal part:~~
~~Photon polarization along momentum axis~~

Transverse part: $\omega = \mathbf{q}$
Photon polarization transverse to momentum axis

Photoabsorption

Recap:



$$R_T(\omega = \mathbf{q}) \rightarrow |\langle \Psi_f | J_T(q) | \Psi_0 \rangle|^2 = \sum_{\lambda=\pm 1} |\langle \Psi_f | J_\lambda(q) | \Psi_0 \rangle|^2$$

Now we can use the multipole decomposition of the current that we just derived

Current operator

General multipole decomposition of the current

$$\mathbf{e}_\lambda \cdot \mathbf{J}\left(\mathbf{q}\right) = (-)^\lambda \sqrt{2\pi(1+\delta_{\lambda 0})} \sum_J \hat{J} \left[L^{el}_{J\lambda}(q)\delta_{\lambda 0} + \left(T^{el}_{J\lambda}(q) + \lambda T^{mag}_{J\lambda}(q) \right) \delta_{|\lambda|1} \right]$$

Real photons: no longitudinal polarization possible
only transverse polarization $\lambda = \pm 1$

$$\mathbf{e}_\lambda \cdot \mathbf{J}\left(\mathbf{q}\right) \longrightarrow (-)^\lambda \sqrt{2\pi} \sum_J \hat{J} \left[\left(\underset{\underset{j_J(qr)}{\uparrow}}{T^{el}_{J\lambda}(q)} + \lambda \underset{\underset{j_{J+1}(qr)}{\uparrow}}{T^{mag}_{J\lambda}(q)} \right) \delta_{|\lambda|1} \right]$$

See when explicitly calculating multipole of a current operator

Low momentum transfer:
Only lowest multipoles prevail J=1 and electric multipole dominates over magnetic

$$J_\lambda(q) \longrightarrow T^{el}_J \xrightarrow{\text{Siegert}} C_{1\pm 1} \rightarrow Y^1(\hat{r}) j(qr) \xrightarrow{\text{low q}} Y^1(\hat{r})\ qr \ \rightarrow \omega \mathbf{r}$$

Photoabsorption

Thus, photoabsorption at low energy can be calculated simply from a dipole response function

$$\sigma(\omega) = \frac{4\pi^2\alpha}{2J_0+1}\omega R(\omega)$$

$$R(\omega) = \sum_f |\langle\Psi_f| D_z |\Psi_0\rangle|^2 \, \delta(E_f - E_0 - \omega)$$

$$D_z = \sum_i^A z_i \left(\frac{1+\tau_i^z}{2}\right)$$

We will see several few- and many-body applications after we have introduced such techniques to calculate wave functions and reactions

Photoabsorption

Comparing calculations in which one uses the dipole operator (Siegert theorem)

$$|\langle \Psi_f | D_z | \Psi_0 \rangle|^2$$

with calculations where one explicitly insert the transverse current (1-body + 2-body, ect.)

$$\sum_{\lambda=\pm 1} |\langle \Psi_f | J_\lambda(q) | \Psi_0 \rangle|^2$$

You see that using the one-body current only it is not enough, whereas using the Siegert theorem it is correct.

Practical Example ^{2}H

From S. Bacca and S. Pastore, J. Phys. G: Nucl. Part. Phys. **41** 123002 (2014).

Work by Pisa and Trento groups