Electromagnetic properties of nuclei: from few- to many-body systems

Lecture 6

Few-body methods

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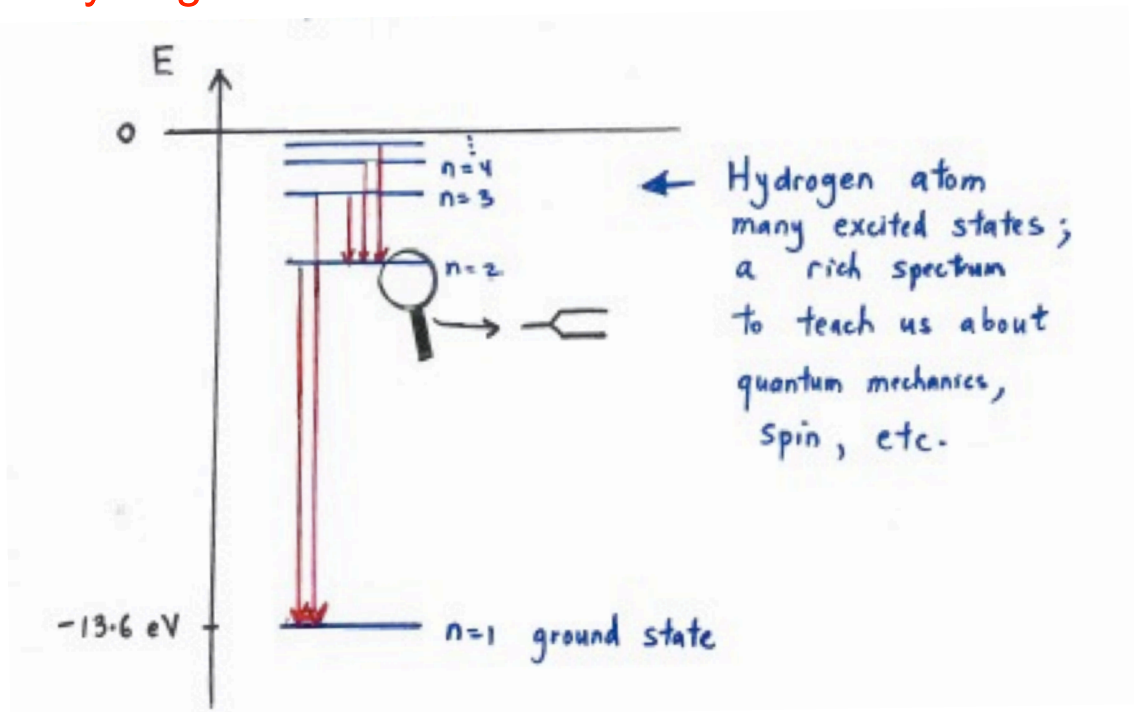
We will worm up with the deuteron,
Which is also the first nuclear few-body system

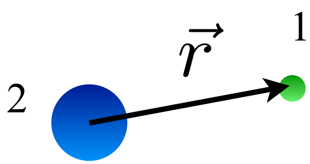
Recap for most of you

The deuteron

- The deuteron ^{2}H (or just d), is the simplest nucleus. It consists of one proton and one neutron. It should be for nuclear physics what the hydrogen atom is to atomic physics.

Hydrogen atom



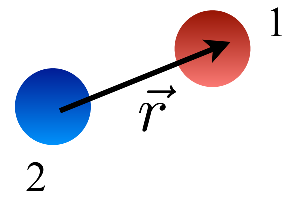


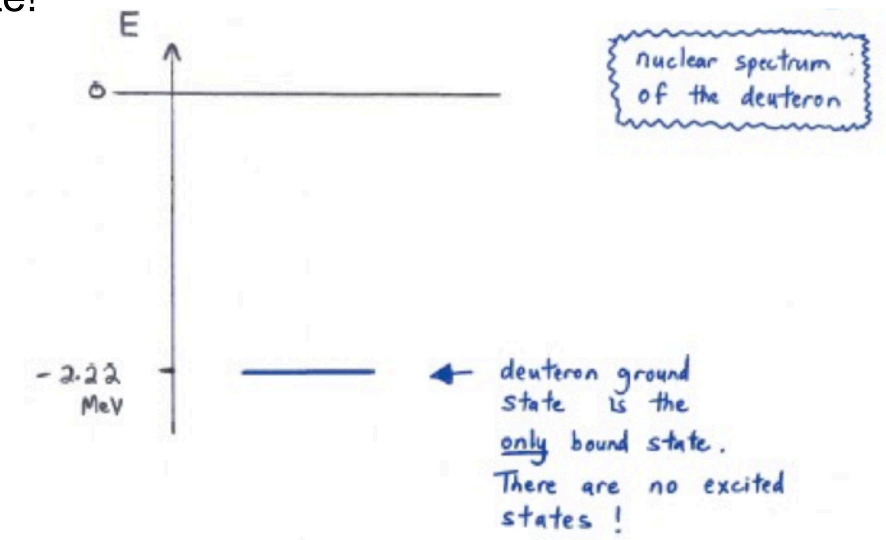
Just as the study of the Balmer series in the electromagnetic transition between the excited stats of the hydrogen has lead to an understanding of its structure, one would like to do the same for nuclear physics and the nuclear force, by studying the deuteron.

The deuteron

The two-body nucleus

However, the deuteron is the only two nucleon bound state, and it has no bound excited state!





For for most nuclei, the typical case is that BE/A ~ 8 MeV.

The deuteron is a very weakly bound nucleus since for it we have BE/A=1.1 MeV.
The deuteron also has a magnetic moment, and a nonzero quadruple moment:

$$\mu = 0.8574 \ \mu_N$$

$$Q = 0.28570 \ \mathrm{fm}^2$$

As we shall see, this means the deuteron is not a pure S-wave state

The deuteron

In the following, we would like to describe the deuteron theoretically.
Let us recall how we want to represent its wave function.

$$|\Psi\rangle = \underbrace{|\psi\rangle}_{space} \otimes \underbrace{|\chi^S\rangle}_{spin} \otimes \underbrace{|\chi^T\rangle}_{isospin}$$

the total wave function must be antisymmetric with respect to the exchange of two p-n

Spin wave function

If we start from two particles with spin 1/2 and want to construct two body states that have good symmetry properties with respect to the permutation group. As you know from quantum mechanics, these are states you can construct

$$\textsc{Triplet}: \begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \end{cases} = \begin{cases} |S=1, S_z=+1\rangle \\ |S=1, S_z=0\rangle \\ |S=1, S_z=-1\rangle \end{cases}$$ symmetric

$$\textsc{Singlet}: \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = |S=0, S_z=0\rangle$$ anti-symmetric

where

$$S_z = S_{z_1} + S_{z_2}$$

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

The deuteron

The same is true for the isospin where the notation is similar, but with a different meaning:

$$|\uparrow\rangle^{iso} = p, \quad |\downarrow\rangle^{iso} = n$$

So we will have singlet and triplet w.f. in isospin as well.

The isospin of the deuteron is T=0, so its isospin wave function must be the isospin singlet which is anti-symmetric

$$|\chi^T\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle^{iso} + |\downarrow\uparrow\rangle^{iso}\right) = |T=0, T_z=0\rangle$$

The deuteron

Can we tell anything more about the spin wave function?
In principle the spin wave function can be either singlet or triplet.

$$\vec{S} = \frac{\vec{1}}{2} + \frac{\vec{1}}{2} = \begin{cases} 0 \\ 1 \end{cases}$$

What we know from experiment is that the total angular momentum of the deuteron is $J^{\pi}=1^{+}$, where the total angular momentum is $\vec{J} = \vec{\ell} + \vec{S} \implies \vec{\ell} = \vec{J} - \vec{S}$

From the above coupling and knowing that J=1, we can get some restrictions on the orbital angular momentum ℓ

If $S = 0 \implies J = \ell = 1$

If $S = 1 \implies 0 = |J-1| \leq \ell \leq J+1 = 2 \implies \ell \in \{0,1,2\}$

All together we have three different possibilities for the orbital angular momentum: 0,1,2

The deuteron

However, we also know that the parity is positive. The parity of the wave function goes like

$\pi = \pi_1 \pi_2 (-1)^\ell = 1$ where π_1, π_2 are the parities of p and n, which are taken to be positive by convention

$\Rightarrow$ $\ell = 1$ cannot happen for the deuteron

So we have that $\cancel{S = 0}$

$$S = 1 \text{ and } \ell \in \{0, 2\}$$

This implies there is an S-wave component $\ell = 0$ (dominant) and a D-wave component $\ell = 2$ in the deuteron w.f.

$$\boxed{\psi = a\,\psi_{(\ell=0)} + b\,\psi_{(\ell=2)} \text{ with } |a|^2 + |b|^2 = 1}$$

The deuteron

Now we want to theoretically solve the deuteron as a two-body problem of particles interacting with a translationally invariant potential in a non relativistic formalism

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V\left(\vec{r_1} - \vec{r_2}\right)$$

If one transforms from particle coordinates to relative and centre of mass coordinates one can simplify the problem to a one-body problem

(we now take the masses $m_1 = m_2 = m$)

$$\begin{cases} \vec{r} = \vec{r}_1 - \vec{r}_2 \\ \vec{R}_{CM} = \frac{\vec{r}_1 + \vec{r}_2}{2} \end{cases} \quad \text{also} \quad \begin{cases} \vec{p} = (\vec{p}_1 - \vec{p}_2)/2 \\ \vec{P}_{CM} = \vec{p}_1 + \vec{p}_2 \end{cases}$$

One obtains $H = \frac{P_{CM}^2}{2M} + \frac{p^2}{2\mu} + V\left(\vec{r}\right) = T_{CM} + H^{\text{int}}$

where $M = 2m, \;\; \mu = \frac{m}{2}$

Hamiltonian depending only on the relative coordinates

The deuteron

We are interested only in the intrinsic dynamics and we do not care if the deuteron moves as a whole with kinetic energy T_{CM}

The Schrödinger equation we want to solve is then:

$$\left[\frac{p^2}{2\mu} + V\left(\vec{r}\right)\right]\psi\left(\vec{r}\right) = E\,\psi\left(\vec{r}\right)$$

Which is clearly a one-body problem in a three-dimensional space.

Go to spherical coordinates and write the laplacian (p^2) in these coordinates

$$\boxed{\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{\hat{\ell}^2}{r^2}\right) + V\left(\vec{r}\right)\right]\psi\left(r,\theta\varphi\right) = E\,\psi\left(r,\theta\varphi\right)} \quad (\star)$$

This is the Schrödinger equation we have to solve in order to obtain a theoretical description of the deuteron.

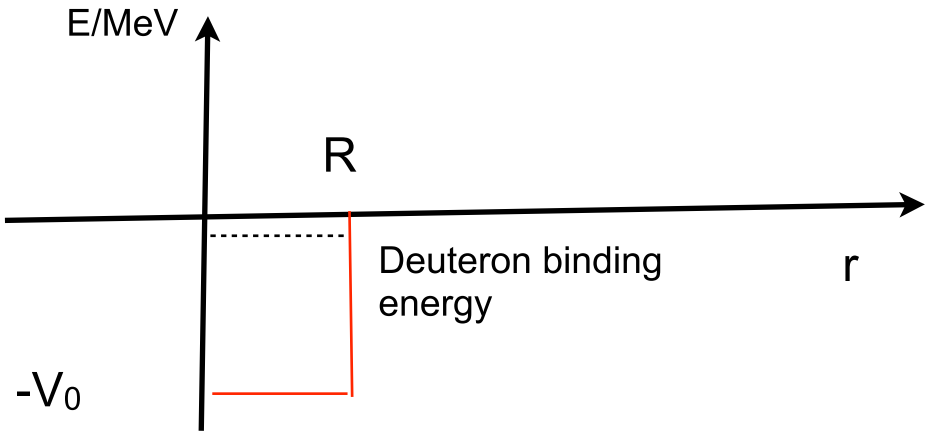
Spherical square-well *analytical*

We first would like to find a simplified solution to the deuteron.

$$\boxed{\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}-\frac{\hat{\ell}^2}{r^2}\right)+V(\vec{r})\right]\psi(r,\theta\varphi)=E\,\psi(r,\theta\varphi)} \qquad (\star)$$

To solve this equation we have to make an ansatz for the potential.

We can find an analytical solution to this equation only if we assume the potential to be very simple: like a spherical square well



$$V(r)=\begin{cases}-V_0 & \text{for } r\leq R\\ 0 & \text{for } r>R\end{cases}$$

In this way the potential depends only on $r=|\vec{r}\,|$ so it is purely central, orbital angular momentum is a good q.n. (We can use ℓ, m, s, m_s as good q.n.)

Spherical square-well *analytical*

Consequently $\Psi(r,\theta,\varphi) = \sum_\ell c_\ell \, R_\ell(r) \, Y_{\ell,m}(\theta,\varphi)$

The g.s. will have $\ell = 0$ $\Psi(r,\theta,\varphi) \propto R_{\ell=0}(r) = R(r)$

Now if you introduce the modified wave function $R(r) = u(r)/r$

then the laplacian in the Schrödinger equation becomes easier in terms of u(r)

$$\boxed{-\frac{\hbar^2}{2\mu}\frac{d^2u(r)}{dr^2} + V(r)u(r) = Eu(r)} \quad (☆) \qquad \text{radial equation}$$

The problem looks now like the a one dimensional spherical potential well that you must have solved analytically in a Quantum Mechanics course. Remember that E is negative for bound states.

For r < R $V = -V_0$ (☆) becomes $-\frac{\hbar^2}{2\mu}\frac{du^2}{dr^2} - V_0 u = Eu$ ➡

$$\frac{d^2u}{dr^2} = -\frac{2\mu(E+V_0)}{\hbar^2}u = -k_1^2 u \qquad \text{with} \qquad k_1 = \sqrt{(2\mu(E+V_0)/\hbar^2)}$$

The solution is sinusoidal w.f. $\boxed{u(r) = A\sin(k_1 r)} + B\cos(k_1 r)$

If we want $R(r)$ to be finite in r=0 then B=0

Spherical square-well *analytical*

For r > R $V = 0$ ($\star$) becomes the free Schrödinger Eq.

$$\frac{d^2u}{dr^2} = -\frac{2\mu E}{\hbar^2}u = k_2^2 u$$ with $$k_2 = \sqrt{-\frac{2\mu E}{\hbar^2}}$$

Remember that E is negative, so this is real

The solution is

$$\boxed{u(r) = Ce^{-k_2 r}} + De^{+k_2 r}$$

To keep this finite at infinite distance we have to require that D=0

Now applying the continuity condition for u and its derivative in r=R we get

w.f. $A\sin(k_1 R) = -Ce^{-k_2 R}$

derivative $k_1 A\cos(k_1 R) = -k_2 Ce^{-k_2 R}$

$$\boxed{k_1 \cot(k_1 R) = -k_2}$$

this gives a relation between V_0 , E and R

Spherical square-well *analytical*

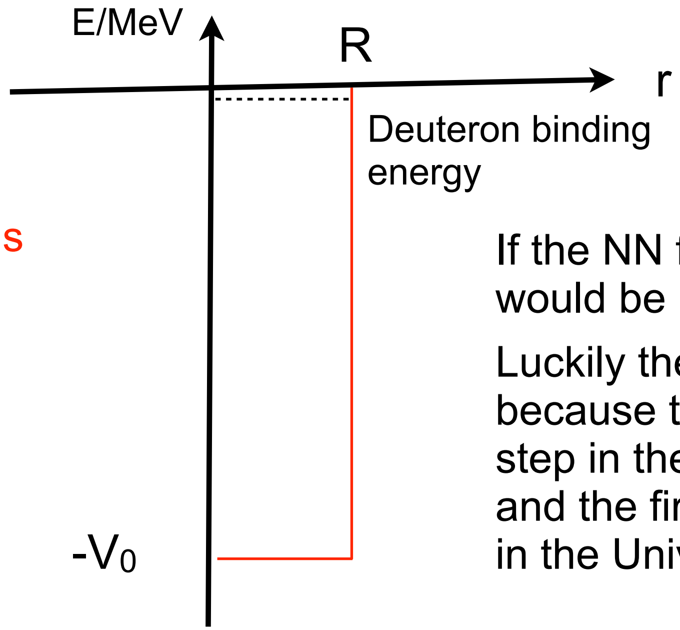
From electron scattering we know R~2.1 fm.

Experimentally we also know that E=-2.2 MeV

So we solve for V_0

$$\boxed{k_1 \cot(k_1 R) = -k_2}$$

It turns out that $V_0 = 35$ MeV



The deuteron is barely bound!

If the NN force were slightly less attractive there would be no bound state, so no deuteron.

Luckily the NN force is attractive enough, because the formation of the deuteron is the first step in the proton-proton fusion cycle in our Sun and the first step in the formation of stable matter in the Universe.

Spherical square-well *analytical*

The weak binding of the deuteron translates into the wave function just barely able to turn over to match the exponential free solution outside the well.

(If the potential were more attractive, the w.f. would turn over earlier).

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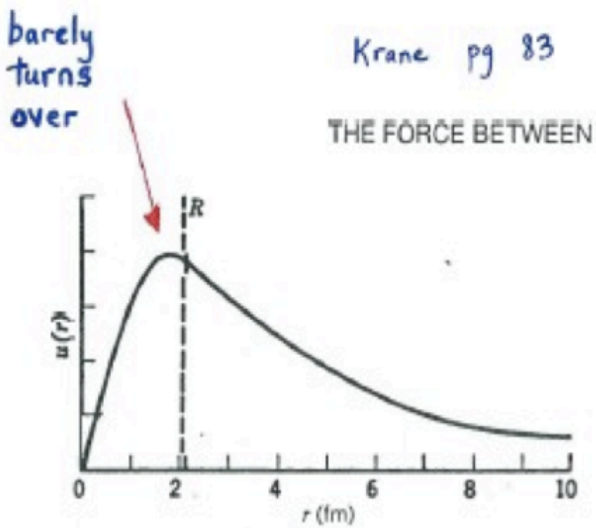


Figure 4.2 The deuteron wave function for $R = 2.1$ fm. Note how the exponential joins smoothly to the sine at $r = R$, so that both $u(r)$ and du/dr are continuous. If the wave function did not "turn over" inside $r = R$, it would not be possible to connect smoothly to a decaying exponential (negative slope) and there would be no bound state.

The deuteron ground state is so close to the top of the well that its wave function leaks way out (extended nucleus).

However …

The truth is that the nuclear potential is not central because

- There is a **spin-orbit** component.
- There is a **tensor force**.

This means one has to couple orbital angular momentum with spin to introduce the total angular momentum J.

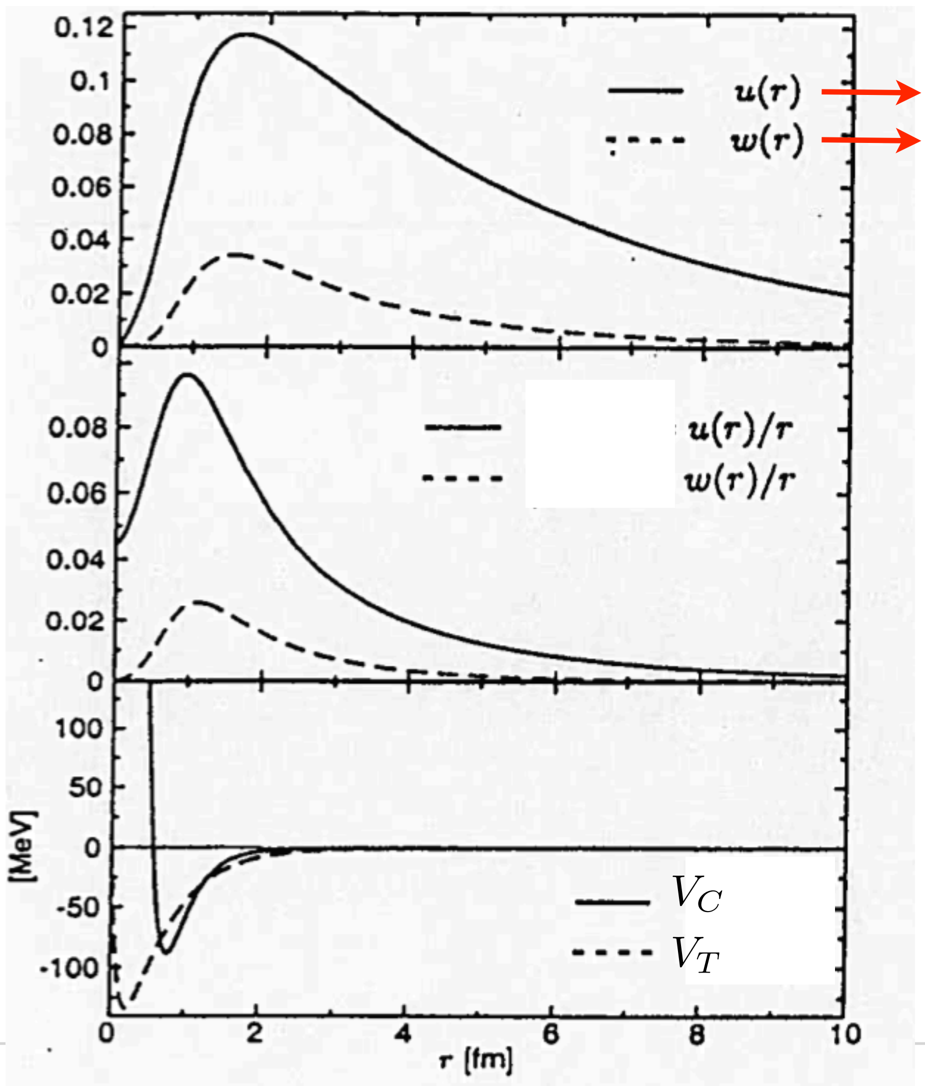
A set of operators that commute with H and have common eigenvalues are J, J_z, S^2 and L^2 .

Main point: Non central forces mix different orbital angular momentum components.
Thus, as we saw before, knowing that J=1, making simple considerations of spin and isospin wave functions + parity, we learn that the deuteron ground state is:

$$\boxed{\psi = a\,\psi_{(\ell=0)} + b\,\psi_{(\ell=2)} \text{ with } |a|^2 + |b|^2 = 1 \quad \text{and} \quad a \gg b}$$

Realistic deuteron

Using a realistic potential, the solution of the Schrödinger equation is not analytical anymore. From a **numerical solution** of the problem one obtains



S-wave $\ell = 0$

D-wave $\ell = 2$

Deuteron properties

Even if we did not know about the one-pion exchange potential, we would realize that the spherical square well is too simple of a model for the deuteron by trying to describe its measured properties with it: we would fail!

- Magnetic Moment $\mu_d = 0.8574 \ \mu_N$
- Quadrupole Moment $Q = 0.285783(30) \ \mathrm{fm}^2 \longleftarrow$ Tensor force

Magnetic moment

Magnetic Moment $\mu_d = 0.8574\ \mu_N$

Even if you do not solve the deuteron numerically, you can tell that a>>b from the magnetic moment.

If there is no orbital angular momentum between p and n (only S-wave component), then magnetic moment of the deuteron is given by

$$\mu_d \approx \mu_p + \mu_n = \left(g_{s_p} S_p + g_{s_n} S_n\right) \mu_N$$

where $g_{s_p} = 5.58569,\ g_{s_n} = -3.82608$

are the anomalous giromagnetic factors of proton and neutron, while $S_{p/n}$ are their spins.

$$\mu_d \approx \left(\frac{5.58569}{2} - \frac{3.82608}{2}\right) \mu_N = 0.879805 \mu_N$$

This is almost correct, but it is not exact. This means that the S-wave component is most of the wave function, but there has to be a small D-wave component to explain this little difference.

Magnetic moment

Magnetic Moment $\quad \mu_d = 0.8574\ \mu_N$

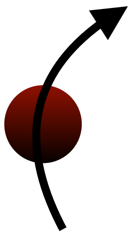
How do you calculate the magnetic moment of the deuteron?

Given an electromagnetic current in coordinate space, the magnetic moment is calculated from the latter as expectation value of this operator

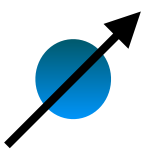
$$\boldsymbol{\mu} = \frac{1}{2}\int d^3x\ \mathbf{x}\times\mathbf{J}(\mathbf{x})$$

Take the convection and spin currents

$$\mathbf{J}^c_{(1)}(\mathbf{x}) = \frac{e}{2m}\sum_i^A \frac{1+\tau_i^z}{2}\{\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_i)\}$$



$$\mathbf{J}^s_{(1)}(\mathbf{x}) = i\frac{e}{2m}\sum_i^A \mu_i \frac{1+\tau_i^z}{2}\boldsymbol{\sigma}_i \times [\mathbf{p}_i, \delta(\mathbf{x}-\mathbf{r}_i)]$$



$$\boldsymbol{\mu} = \sum_i^A \mu_N \left[\left(\frac{\mu^S+\mu^V\tau_i^z}{2}\right)\overset{\mathbf{S}_i}{\boldsymbol{\sigma}_i} + \left(\frac{1+\tau_i^z}{2}\right)\boldsymbol{\ell}_i\right]$$

LO in chiral EFT

Magnetic moment

Magnetic Moment $\mu_d = 0.8574\ \mu_N$

Take the z component, write it for 2 nucleons only and use the isoscalar and isovector anomalous moments

$$\mu_d = \mu_N \sum_{i=1}^{2} \left(\underbrace{[4.7\tau_i^z + 0.88]}_{g_s} S_{z_i} + \underbrace{\frac{1+\tau_i^z}{2}}_{g_\ell} \ell_{z_i} \right)$$

where

$$g_s|p\rangle = [4.7\tau^z + 0.88]|p\rangle = [4.7 + 0.88]|p\rangle = 5.58|p\rangle$$

$$g_s|n\rangle = [4.7\tau^z + 0.88]|n\rangle = [-4.7 + 0.88]|n\rangle = -3.82|n\rangle$$

$$g_\ell|p\rangle = \frac{1+\tau^z}{2}|p\rangle = 1|p\rangle$$

$$g_\ell|n\rangle = \frac{1+\tau^z}{2}|n\rangle = 0$$

There always is spin part and an orbital part

Magnetic moment

Magnetic Moment $\mu_d = 0.8574\ \mu_N$

$$\boxed{\mu_d = \mu_N \sum_{i=1}^{2} \left([4.7\tau_i^z + 0.88] S_{z_i} + \frac{1+\tau_i^z}{2} \ell_{z_i} \right)}$$

Expectation value has to be taken on the total w.f.

If a general operator $O = O^{spin-space} \otimes O^{isospin}$

acts on different part of the space $|\Psi\rangle = \underbrace{[\,\psi^L \otimes \chi^S]^J\rangle}_{spin-space} \otimes \underbrace{|\chi^T\rangle}_{isospin}$

In the expectation value each part of the operator acts only on the corresponding w.f.

$$\langle\Psi|O|\Psi\rangle = \langle[\psi^L \otimes \chi^S]^J|O^{spin-space}|[\psi^L \otimes \chi^S]^J\rangle \boxed{\langle\chi^T|O^{isospin}|\chi^T\rangle}$$

Magnetic moment

Magnetic Moment $\mu_d = 0.8574 \ \mu_N$

$$\boxed{\mu_d = \mu_N \sum_{i=1}^{2} \left([4.7\tau_i^z + 0.88] S_{z_i} + \frac{1+\tau_i^z}{2} \ell_{z_i} \right)}$$

Isospin part

$\boxed{\tau_i^z}$ This part is a vector in spin space (isovector). The deuteron has T=0, so when you sandwich this with the wave function you get 0

$$\langle T=0 | \, \tau_1^z + \tau_2^z \, | T=0 \rangle = 0$$

$\boxed{\mathbf{1}}$ For the isoscalar part you get

$$\langle T=0 | \mathbf{1} | T=0 \rangle = 1$$

We can just take the isoscalar part of the operator

Magnetic moment

Magnetic Moment $\mu_d = 0.8574\ \mu_N$

$$\mu_d = \mu_N \left(0.88 \underbrace{\sum_{i=1}^{2} S_{z_i}}_{S_z} + \frac{1}{2} \underbrace{\sum_i \ell_{z_i}}_{\ell_z} \right)$$

NB: $\ell_{z_1} + \ell_{z_2} = \ell_z + L_z^{CM}$

we can drop the CM part

When we take the expectation value on the spin-space we have

$$\langle \mu_d \rangle = \mu_N \left(0.88 \langle S_z \rangle_{M=1} + \frac{1}{2} \langle \ell_z \rangle_{M=1} \right)$$

NB: Magnetic moment is defined as expectation value on state with maximally alligned total angular momentum, i.e. J,Jz=1,1 for the deuterium

Because $J_z = \ell_z + S_z \implies \ell_z = J_z - S_z$ and $\langle J_z \rangle_{M=1} = 1$

We have $\langle \ell_z \rangle_{M=1} = \langle J_z \rangle_{M=1} - \langle S_z \rangle_{M=1} = 1 - \langle S_z \rangle_{M=1}$

$$\boxed{\langle \mu_d \rangle = \mu_N \left(0.38 \langle S_z \rangle_{M=1} + \frac{1}{2} \right)}$$

Magnetic moment

Magnetic Moment $\mu_d = 0.8574\ \mu_N$

$$\langle \mu_d \rangle = \mu_N \left(0.38 \langle S_z \rangle_{M=1} + \frac{1}{2} \right)$$

In calculating the expectation value of S_z we have to consider that

$$\psi = a\psi_{(\ell=0,S=1,J=1)} + b\psi_{(\ell=2,S=1,J=1)}$$

It can be shown that $\langle S_z \rangle = 1\ on\ \psi_{(\ell=0,S=1,J=1)}$

$$\langle S_z \rangle = -1/2\ on\ \psi_{(\ell=2,S=1,J=1)}$$

So if a=1, there is a pure S-wave and $\langle \mu_d \rangle = \mu_N 0.88$

And if b=1, there is a pure D-wave $\langle \mu_d \rangle = \mu_N 0.31$

Magnetic moment

Magnetic Moment $\mu_d = 0.8574\ \mu_N$

Because S-wave and D-wave states are not coupled by the operator, we can write that in general

$$\langle \mu_d \rangle = \mu_N \left(0.88\, a^2 + 0.31\, b^2\right)$$

with $a^2 + b^2 = 1 \implies a^2 = 1 - b^2$

Then $\langle \mu_d \rangle = \mu_N \left(0.88 - 0.527\, b^2\right) \equiv 0.8574 \mu_N$

Linear relation between D-wave probability (b^2) and magnetic moment

Impose it to be equal to the experimental value

$b^2 = 0.04$ 4% of D-wave in the deuteron wave function

Magnetic moment

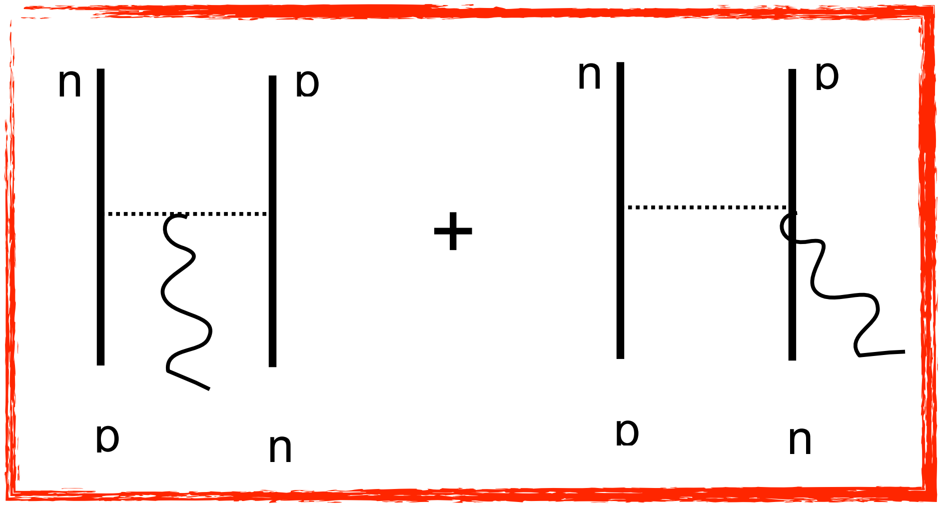
Magnetic Moment $\mu_d = 0.8574 \ \mu_N$

N.B.: This is not the whole story. A more sophisticated theoretical treatment shows that our starting magnetic operator is not complete, because there are meson exchange currents.

With these two-body operators then b^2 becomes 0.05-0.07.

Chiral EFT

Magnetic Moment $\mu_d = 0.8574\ \mu_N$



Leading term of two-body

They appear at NLO but are purely isovector, so they do not contribute to the deuteron magnetic moments.

Loop corrections and other higher order terms contribute and contain new LEC which can be calibrated by fitting the experimental value.

EM observables are needed to fit the new LEC in the em operators