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Stellar Evolution Lecture week

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Problem Sheet 1

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1. Starting from the equation of the mass conservation in spherical symmetry and the fact that
$$\vec{\boldsymbol{g}} = -\nabla V = -\frac{GM_r}{r^2}\hat{\boldsymbol{r}},$$
where V is the gravitational potential, re-derive the Poisson equation in spherical coordinate: $\Delta V = 4\pi G\rho$.
[In spherical coordinates, the second derivative of a scalar field, A,
$\Delta A = \frac{\partial^2 A}{\partial r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)$]

2. In spherical symmetry, The change of mass, M_r, in a sphere of radius, r, considering gas motion (v_r in spherical coordinates) is:
$$\mathrm{d}M_r(r,t) = (4\pi r^2\rho)\ \mathrm{d}r - (4\pi r^2\rho v_r)\ \mathrm{d}t \qquad (1)$$
Knowing that the second crossed derivatives of a function $f(x,y)$ are equal: $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$, re-derive the continuity equation: $\frac{\partial\rho}{\partial t} + \nabla\rho\vec{\boldsymbol{v}} = 0$.
[In spherical coordinates, the divergence of a vector field $\vec{\boldsymbol{u}}$, $\nabla\vec{\boldsymbol{u}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\mathrm{sin}\theta}\frac{\partial}{\partial\theta}\left(u_\theta\,\mathrm{sin}\theta\right) + \frac{1}{r\mathrm{sin}\theta}\frac{\partial}{\partial\phi}\left(u_\phi\right)$]

3. Show that the dynamical timescale is also equal to $\tau = \frac{R}{c_s}$, where c_s is the speed of sound and is given by $c_s = \sqrt{\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_{ad}}$. [Ignore gravity in the equation of motion].

4. Show that the following relation holds between the gravitational potential V and the gravitational energy, Ω:
$$\Omega = \frac{1}{2}\int_0^M V\mathrm{d}M_r$$
[$-\Omega = G\int_0^M \frac{M_r\mathrm{d}M_r}{r}$]

5. Show that the total potential energy for a spherical star with constant density is:
$$\Omega = -q\frac{GM^2}{R},$$
and find the value of q.

6. (a) Estimate the fraction of the mass of protons that is converted to energy during hydrogen burning.

(b) Considering that 10% of the mass of the Sun takes part in nuclear fusion, establish a formula for the nuclear lifetime of hydrogen burning for the Sun.

(c) Repeat part (a) for helium and carbon burning. How does this impact the lifetime of these burning stages compared to hydrogen burning. Discuss the implications.

7. (a) Using the mass-luminosity relation, establish a formula for the lifetime of a star of mass, $M > 1M_\odot$ relative to the lifetime of the Sun. Apply your formula to stars 10 and 100 times the mass of the Sun. Discuss your answers.

   (b) Calculate the dynamical and KH timescales for stars of 1, 10 and 100 $M_\odot$.

   (c) The mass-luminosity relationship is actually almost linear above 20 $M_\odot$. Discuss how this changes your results.

The end.