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Stellar Evolution Lecture week

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Problem Sheet 3

1. In this exercise, we will solve "numerically" (by hand actually), the one-dimensional (1D) diffusion equation:

$$\partial_t Y(x,t) = \nu \partial_x^2 Y(x,t),$$

for a quantity Y. We assume that the diffusion occurs in water ($\nu = 1 \cdot 10^{-6}\,\mathrm{m^2/s}$). We consider a pool of 0.1-m length. The initial distribution of the quantity $Y(x,t)$ is given by (see Fig. 1):

$$Y(x,t=0) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

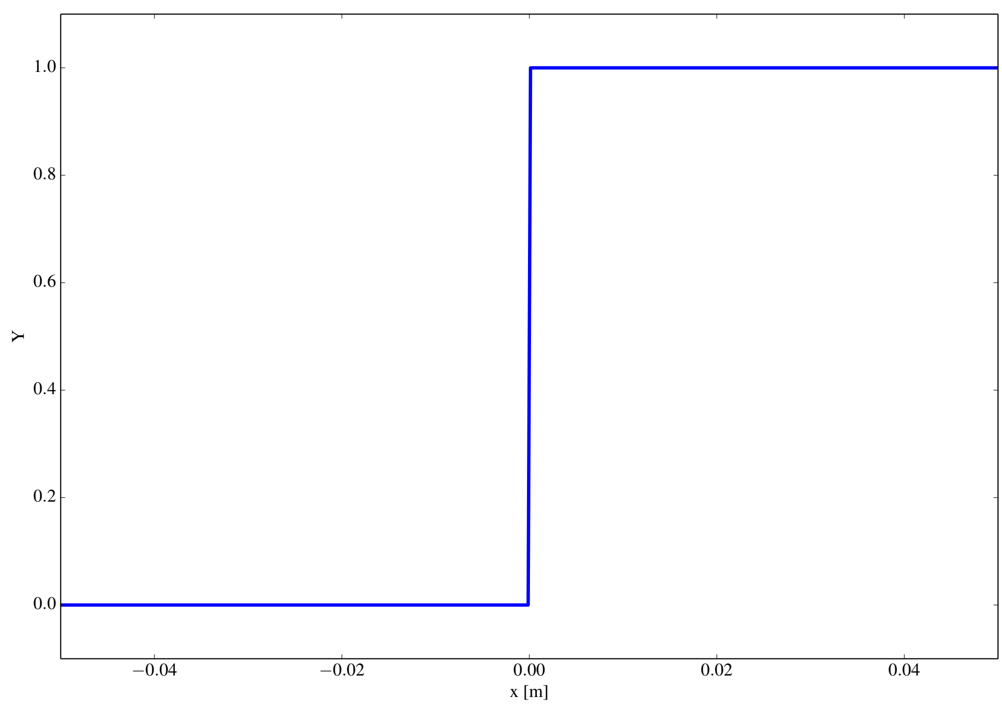

Figure 1: Initial distribution of the quantity $Y(x,t)$ in the pool.

We will consider 10 cells of the same size along the x-axis, between the points $x = -0.05$ and $x = 0.05$. The coordinates of the edge of the cells are:

$$x_{\mathrm{edge}} \in (-0.05, -0.04, -0.03, -0.02, -0.01, 0, 0.01, 0.02, 0.03, 0.04, 0.05)$$

(a) In the following, we will consider that the quantity Y is defined at the centre of each cell. We have thus to know the coordinate of the centre of the 10 cells. Determine these coordinates and draw these in Fig. 1.

(b) With this spatial discretisation, what is the maximal time-step Δt_{max} that is allowed by the CFL (in)stability condition? Obviously, this is the limit, above which we are sure that the scheme is unstable. However, it does not tell us if a smaller time step is stable. To be safe, we will use a time step $\Delta t = \frac{\Delta t_{\text{max}}}{10}$. [$\Delta t_{\text{max}}^{\text{CFL}} = \frac{\Delta x^2}{\nu}$.]

(c) With the definition of the +-midpoint (right-hand-side) discretised derivative of the quantity Y:

$$\hat{\nabla}_x^+ Y = \frac{Y(x+\Delta x) - Y(x)}{\Delta x},$$

write an expression for the discretised double derivative $\hat{\nabla}_x^2$.

(d) Let us consider a given time-step t_j. For this time step, and for a given cell x_i, write the discretised version of the diffusion equation (using the notation Y_i^j to indicate the value of $Y(x_i, t^j)$ in the cell i (centred at the coordinate x_i) at the time t^j).

(e) Re-write the expression you just found to express Y_i^{j+1} as a function of the quantities at time-step j: Y_i^j, Y_{i+1}^j and Y_{i-1}^j.

(f) Starting from our initial conditions, compute $\frac{\nu \Delta t}{\Delta x^2}\left(Y_{i+1}^0 + Y_{i-1}^0 - 2Y_i^0\right)$.

(g) Then compute the new values of Y at the next time step. Sketch the results in Fig. 1.

(h) Repeating this procedure, compute the evolution of the quantity Y over a few time-steps, and plot the results.

2. A velocity-driven planar flow is governed by the following diffusion equation:

$$\partial_t v_x = \nu \partial_y^2 v_x.$$

Consider a horizontal plate of infinite area in the x-z plane (y is the vertical axis), immersed in an infinite sea. The plate oscillates in the x-direction at a frequency ω, with the following velocity:

$$U(t) = U_0 \cos(\omega t),$$

with a constant amplitude U_0.

(a) Show that the velocity field given by:

$$v_x(y,t) = U_0 \exp(-ky)\cos(\omega t - ky),$$

where k is the wave number, is a solution of the diffusion equation and obtain a relation between k, ω and ν.

(b) As k appears in the exponential as well, $d = \frac{1}{k}$ is a characteristic decay length of the wave. For kinematic viscosity, $\nu_{\text{water}} = 1.0 \cdot 10^{-6}\,\text{m}^2\text{s}^{-1}$, and for a frequency of $f_{\text{plate}} = 500\,\text{Hz}$, calculate d.