Darmstadt Lecture 6 – Supernova Collapse Models

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Neutrino Opacities

In the Fe cores of massive progenitor stars prior to gravitational collapse, the electron fraction is $Y_e = Z/A \simeq 0.42$. During collapse, Y_e decreases further, and ultimately will reach values near $Y_e \simeq 0.04$ in the final neutron star. However, the short mean free path of neutrinos in dense matter delays this reduction.

The weak interaction cross section is

$$\sigma_o = 4\pi \left(\frac{m_e c}{\hbar}\right)^4 \left(\frac{G_F}{m_e c^2}\right)^2 = 1.76 \cdot 10^{-44} \text{ cm}^2.$$

The four major neutrino-matter interactions are

Neutral current free nucleon scattering $(\nu + n \xrightarrow{Z} \nu + n, \nu + p \xrightarrow{Z} \nu + p)$

$$\sigma_n = \frac{\sigma_o}{4}\left(\frac{E_\nu}{m_e c^2}\right)^2 = 1.7 \cdot 10^{-44} \frac{E_\nu^2}{\text{MeV}^2} \text{ cm}^2 \qquad \text{ND}$$

$$\sigma_n = \frac{\pi^2 \sigma_o}{64}(1+2g_A^2)\left(\frac{k_B T}{m_e c^2}\right)^2 \frac{E_\nu}{p_F c}\frac{m_n c^2}{\epsilon_F} = 2.1 \cdot 10^{-45} \frac{T^2 E_\nu}{\text{MeV}^3}\frac{n_s}{n} \text{ cm}^2 \qquad \text{ED}$$

Neutral current heavy nucleus scattering ($\nu + (Z, A) \xrightarrow{Z} \nu + (Z, A)$),

$$\sigma_A = \frac{\sigma_o}{16}\left(\frac{E_\nu}{m_e c^2}\right)^2 \left[A + Z(4\sin^2\theta_W - 2)\right]^2 \simeq 4.2 \times 10^{-45} N^2 \frac{E_\nu^2}{\mathrm{MeV}^2}\ \mathrm{cm}^2$$

Charged current nucleon absorption ($\nu_e + n \xrightarrow{W} p + e^-, \bar{\nu}_e + p \xrightarrow{W} n + e^+$)

$$\sigma_a = \frac{\sigma_o}{2}(1 + 3g_a^2) Y_e \left(\frac{E_\nu}{m_e c^2}\right)^2 = 1.9 \cdot 10^{-43} Y_e \frac{E_n u^2}{\mathrm{MeV}^2}\ \mathrm{cm}^2 \qquad \mathrm{ND}$$

$$\sigma_a = \frac{3\pi^2 \sigma_o}{128}(1 + 3g_A^2)\left(\frac{k_B T}{m_c c^2}\right)^2 \frac{m_n c^2}{\epsilon_F}\left(\frac{Y_e}{1 - Y_e}\right)^{1/3}$$

$$= 1.43 \cdot 10^{-42}\left(\frac{Y_e}{1 - Y_e}\right)^{1/3} \frac{T^2}{\mathrm{MeV}^2}\left(\frac{n_s}{n}\right)^{2/3}\ \mathrm{cm}^2 \qquad \mathrm{ED}$$

Charged and neutral current electron scattering ($\nu + e^- \xrightarrow{W,Z} \nu + e^-$)

$$\sigma_e = 0.1\sigma_o \left(\frac{E_\nu}{M_e c^2}\right)^2 \frac{E_\nu}{\mu_e} \simeq 2.0 \cdot 10^{-47}\left(\frac{n_s}{n Y_e}\right)^{1/3} \frac{E_\nu^3}{\mathrm{MeV}^3}\ \mathrm{cm}^2$$

The largest source of opacity during collapse is from coherent scattering. Neutrino mean free path $\lambda_\nu = <\sigma n>^{-1}$ ($\rho_{12} = n m_n/(10^{12}\ \mathrm{g\ cm}^{-3})$) is

$$\lambda_\nu \simeq \frac{60}{\rho_{12}}\left(6X_n + 5X_p + A(1 - x_N)^2 X_A\right)^{-1}\left(\frac{10\ \mathrm{MeV}}{E_\nu}\right)^2\ \mathrm{km}.$$

Neutrino Trapping

Effectively, λ_ν equals the core's size when

$$\lambda_\nu \simeq \left(\frac{3M}{4\pi\rho}\right)^{1/3} = 87.3\left(\frac{M}{1.4\ M_\odot}\right)^{1/3} \rho_{12}^{-1/3}\ \mathrm{km}$$

which occurs when $\rho \simeq 3 \cdot 10^{10}$ g cm^{-3}.

But trapping only occurs later, when the neutrino diffusion timescale is smaller than the collapse timescale.

Diffusion in spherical symmetry: The flux is driven by a density gradient.

$$F_\nu = -\frac{c\lambda_\nu}{3}\frac{\partial n_\nu}{\partial r}$$

The diffusion equation is:

$$\frac{\partial n_\nu}{\partial t} = -\frac{1}{r^2}\frac{\partial r^2 F_\nu}{\partial r} = \frac{c\lambda_\nu}{3r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial n_\nu}{\partial r}\right].$$

For simplicity, assume λ_ν is a constant, and seek a separable solution of the type $n_\nu = n_o\psi(r)\phi(t)$. We find

$$\frac{1}{\phi}\frac{d\phi}{dt} = \frac{c\lambda_\nu}{3r^2\psi}\frac{d}{dr}\left[r^2\frac{d\psi}{dr}\right] = -\alpha$$

where α is a constant.

Solving each differential equation, we find

$$\phi = \phi_o e^{-\alpha t}, \qquad \psi = \frac{\sin \beta r}{\beta r}, \qquad \beta = \sqrt{\frac{3\alpha}{c\lambda_\nu}}.$$

Note that the radial equation is the same as the Lane-Emden equation for polytropic index $n = 1$. If one takes the radius of the neutrinosphere to be $R \simeq \pi/\beta$, since $\psi = 0$ there, the diffusion timescale is

$$\tau_d = \frac{1}{\alpha} = \frac{3R^2}{\pi^2 c\lambda_\nu} \simeq 0.013\rho_{12} \left(\frac{M}{1.4\ M_\odot} \frac{Y_\nu}{0.06} \right)^{2/3} \mathrm{s}.$$

The collapse timescale can be estimated from self-similar collapse models,

$$\tau_c = f \left(\frac{3}{8\pi G\rho} \right)^{1/2} \simeq 0.0077\rho_{12}^{-1/2}\ \mathrm{s}$$

where $f = \sqrt{153}$ for $\gamma = 4/3$ and $f = \sqrt{33}$ for $\gamma = 1.30$, which we used in the above. Equating these timescales gives the trapping density

$$\rho_{12,trap} \simeq 0.71 \left(\frac{M}{1.4\ M_\odot} \frac{Y_\nu}{0.06} \right)^{-4/9}$$

Entropy During the Collapse

Following trapping, we will find the collapse proceeds approximately adiabatically, since significant heat is unable to be lost on collapse timescales.

The entropy prior to collapse has contributions from nuclei, free neutrons and protons, and electrons. Assume $T \simeq 0.7$ MeV, $\rho \simeq 10^9$ g cm^{-3} and $Y_e \simeq 0.42$.

Nuclear entropy originates from translation and excited states. Per nucleus:

$$\frac{S_{H,trans}}{k_B} = \frac{5}{2} + \ln\left[\left(\frac{56 m_b k_B T}{2\pi\hbar^2}\right)^{3/2}\frac{1}{n_H}\right] \simeq 17, \qquad \frac{S_{H,ex}}{k_B} = 56\frac{\pi^2}{2}\frac{T}{E_F} \simeq 4.8$$

where n_H is the number density of nuclei (assumed to be iron) and $E_F \simeq 35$ MeV is the Fermi energy of nuclear matter.

The electron entropy, per electron, is $S_e = \pi^2 k_B T/\mu_e \simeq 1.1 k_B$.

The dilute vapor of nucleons (mostly neutrons) has an entropy per nucleon

$$\frac{S_{n,p}}{k_B} = \frac{5}{2} + \ln\left[\left(\frac{m_b k_B T}{2\pi\hbar^2}\right)^{3/2}\frac{2}{n_{n,p}}\right] \simeq (12.9, 36)$$

where n_n is the neutron density. The total entropy per baryon is thus

$$s = X_H \frac{S_{H,trans} + S_{H,ex}}{56} + S_e Y_e + S_n X_n + S_p X_p \simeq 0.92 k_B.$$

For comparison, the solar center has $s \simeq 16.5 k_B$.

Thermodynamics During Collapse

The first law of thermodynamics can be written

$$\dot{Q} = k_B T \dot{s} + \sum_i \mu_i \dot{Y}_i = < E_{\nu,esc} > \left(\dot{Y}_e + \dot{Y}_\nu \right),$$

where $i = (H, n, p, e)$. The heat change $\dot{Q}$ is due to escaping neutrinos. In nuclear statistical equilibrium

$$\sum_i \mu_i \dot{Y}_i = \dot{Y}_e(\mu_e - \hat{\mu}) + \dot{Y}_\nu \mu_\nu,$$

so

$$k_B T \dot{s} = -\dot{Y}_e \left(\mu_e - \hat{\mu} - \mu_\nu \right) - \left(\dot{Y}_e + \dot{Y}_\nu \right) \left(\mu_\nu - < E_{\nu,esc} > \right).$$

There is entropy generation from being out of beta equilibrium, and from neutrino energy losses. Early on, neutrinos freely escape, and $\mu_\nu = 0$. When neutrinos are trapped, $< E_{\nu,esc} >= \mu_\nu$ because only neutrinos at the top of the Fermi sea will escape. Thus

$$k_B T \dot{s} = -\dot{Y}_e \left(\mu_e - \hat{\mu} - < E_{\nu,esc} > \right), \quad \text{or} \quad k_B T \dot{s} = -\dot{Y}_e \left(\mu_e - \hat{\mu} - \mu_\nu \right)$$

depending on trapping. Initially, entropy changes, but once trapping ensues and beta equilibrium follows, the entropy is frozen.

Y_e Changes During Collapse

Changes in composition (Y_e) are due to electron captures on heavy nuclei and free protons.

Electron captures on heavy nuclei are very sensitive to the energy available for electron capture, $\Delta = \mu_e - \hat{\mu}$, perhaps as the 3rd or 4th power. As the electron fraction decreases, the heavy nuclei become more neutron rich and more massive. But shell closures will effectively halt electron captures on heavy nuclei some time after the collapse begins.

Following this, electron captures are dominated by those on free protons

$$\dot{Y}_e = \frac{3}{5}\left(\frac{\mu_e}{m_e c^2}\right)^2 n Y_e X_p \sigma_o c \simeq 488 \rho_{12} Y_e X_p \mu_e^2 \ \mathrm{s}^{-1}$$

The free proton abundance is $X_p \propto e^{\mu_p/k_B T}$.

In the liquid drop model, the dominant nucleus has $A = (a_s + S_s I^2)/(2a_c x^2)$;

$$\mu_p = -B + \frac{2}{3}\frac{a_s}{A^{1/3}} + \frac{a_c x A^{2/3}}{3}(6-x) + (1-2x)\left[S_v(2x-3) + \frac{2S_s}{3A^{1/3}}(4x-5)\right].$$

Thus $\partial\mu_p/\partial x \simeq 90$ MeV showing that μ_p is very sensitive to $Y_e \simeq x$. As Y_e decreases, μ_p falls, which decreases both X_p and $\dot{Y}_e$. Therefore, electron captures are highly self-regulating. Y_e falls slowly with increasing density.

Electron captures on nuclei produce neutrinos with average energy $\bar{\epsilon}_\nu \simeq 3\Delta/5$, where $\Delta = \mu_e - \hat{\mu} - 3\mathrm{MeV}$ is the maximum energy available for captures. (3 MeV represents a typical excitation energy in the daughter nucleus.)

Before trapping, when neutrinos escape freely, this leads to an entropy increase $k_B T\dot{s} \simeq -\dot{Y}_e[(2/5)(\mu_e - \hat{\mu}) + 1.8\mathrm{MeV}] \simeq -4\dot{Y}_e\mathrm{MeV}$.

Electron capture on free protons produces neutrinos with a larger $\bar{\epsilon}_\nu \simeq 5\mu_e/6$. Before neutrino trapping, this leads to an entropy loss $k_B T\dot{s} \simeq -\dot{Y}_e(\mu_e/6 - \hat{\mu}) \simeq 9\dot{Y}_e\ \mathrm{MeV}$.

After the density exceeds 10^{12} g cm^{-3}, neutrinos become trapped. Inverse capture reactions build so that $\dot{Y}_\nu \to -\dot{Y}_e$, and the net lepton number $Y_L = Y_e + Y_\nu$ becomes frozen. Although Y_e decreases further with increasing density, Y_ν rises to compensate.

After trapping, neutrinos become degenerate with $\mu_\nu \to \mu_e - \hat{\mu}$ as beta equilibrium finally becomes established, and we have $T\dot{s} \simeq 0$.

There is only a small window early in the collapse where the entropy can change, and changes due to the two modes of electron capture largely cancel. Effectively, the entropy per baryon remains near unity throughout collapse.

Calculations show that at the end of collapse the trapped lepton fraction $Y_L \simeq 0.38$, with $Y_e \simeq 0.32$, whereas initially $Y_e \simeq 0.42$.

Core Bounce and Shock Formation

The inner core collapses homologously and subsonically, and when the central density reaches nuclear densities, the resulting bounce affects the entire homologous core. The velocity gradient at the core's edge steepens into a shock, which moves outwards from the edge of the core, whose interior remains unshocked, at about $r = 30$ km. The shock effectively damps inner core oscillations, which forms a proto-neutron star.

However, the shock has to do work, not only reversing the motion of matter it encounters, but also dissociating nuclei at a cost of nearly 8 MeV per baryon. As a result, the shock at least temporarily stalls when it reaches a distance of $r = 100$ to 200 km.

The energy of the shock originates from binding energy released by the formation of the unshocked core. In general, one expects this binding energy will scale with the dimensions of the inner core as $GM_{ic}^2/R_{ic} \propto M_{ic}^{5/3}$. For polytropes, masses scale as $K^{3/2}$. For matter near n_s, whose pressure is dominated by leptons, we expect $K \propto Y_L^{4/3}$. Thus, the binding energy should scale as $Y_L^{10/3}$.

Core Binding Energy

Below n_s, the effective polytropic exponent is about 4/3, and if the inner core had a sub-nuclear central density, we would expect its total energy to be approximately zero. However, the central density exceeds n_s and the effective polytropic exponent at those densities is much larger due to nuclear repulsion. Indeed, self-similar results for $\gamma \lesssim 4/3$ indicate the core mass will exceed the effective Chandrasekhar mass by about 10%.

The mass in the inner core residing at densities greater than n_s leads to binding, as can be seen by considering nested polytropes. Consider the equation of state satisfying

$$p = K\rho^{1+1/n}; \qquad \varepsilon = np \qquad \rho < \rho_t$$

$$p = K\rho_1^{1/n-1/n_1}\rho^{1+1/n_1}; \qquad \varepsilon = n_1 p + (n-n_1)p_t \qquad \rho > \rho_t$$

For this equation of state, one can manipulate the identities $d\Omega = VdP$ and $dU = \varepsilon dV$, where $E = U + \Omega$, to find

$$E = \frac{n-3}{5-n}\frac{GM^2}{R} + \left[\frac{n_1-3}{5-n_1} - \frac{n-3}{5-n}\right]\frac{GM_t^2}{R_t} + 3P_t\left[\frac{M_t}{\rho_t} - V_t\right]\left[\frac{n-1}{5-n} - \frac{n_1-1}{5-n_1}\right].$$

The mass, radius and volume interior to the point where $\rho = \rho_t$ are M_t, R_t and V_t, respectively. In the idealized case where $n = 3$ and $n_1 = 0$, one sees that $E = -(3/5)GM_t^2/R_t$; the high-density core provides binding.

Prompt Success of the Shock

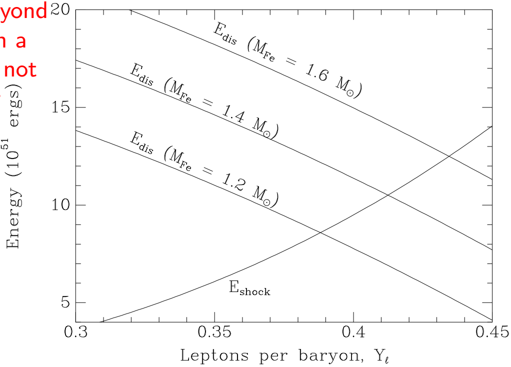
Although one can't immediately estimate M_t or R_t, it is possible to note that in hydrostatic equilibrium, the energy change from adding the mass dM is $dE = -GMdM/R$, so one expeccts the binding energy to be

$$\mathrm{BE} = GM_{Ch}(M - M_{Ch})/R \simeq 0.1GM_{Ch}^2/R_{Ch} \simeq (5-10)\cdot 10^{51}\ \mathrm{erg\ s}^{-1}.$$

As we established earlier, BE $\propto Y_{L,fin}^{10/3}$, with $Y_{L,fin}$ the final lepton fraction.

To lowest order, a successful shock has to be able to dissociate the remainder of the iron core that accretes through the shock onto the core. The rapid steepening of the density gradient beyond the edge of the iron core will result in a successful shock, if only the shock is not carried inward by the infalling matter before it can propagate that far.

Dissociation of iron nuclei comes at a cost of about 9 MeV/nucleon or $1.8\cdot 10^{52}$ erg/$M_\odot$. The amount of mass to be dissociated is the mass of the initial iron core minus the mass of the final homogous core, i.e. $M_{dis} \propto Y_{e,init}^2 - Y_{L,fin}^2$. We have seen that $Y_{e,init} \simeq 0.42$ and $Y_{L,fin} \simeq 0.38$.



Long Term Shock Success

It's long been realized that neutrinos are important in the supernova mechanism. The unshocked core, whose entropy is about $s \sim 1k_B$ per baryon, has $T \simeq 20$ MeV. Within, neutrinos are trapped, but beyond about $R_\nu \sim 30$ km, neutrinos escape with a roughly thermal distribution with a temperature about $T_\nu \sim 4-5$ MeV. The shock has stalled at $R_s \sim 100-200$ km.

The dominant means of exchanging energy between neutrinos and matter is through charged current nucleon absorption, for which the cross section, assuming non-degeneracy, is

$$\sigma_a = \frac{\sigma_o}{2}(1+g_A^2)Y_e\left(\frac{E_\nu}{m_e c^2}\right)^2.$$

With $Y_e = 1/2$, this results in an opacity

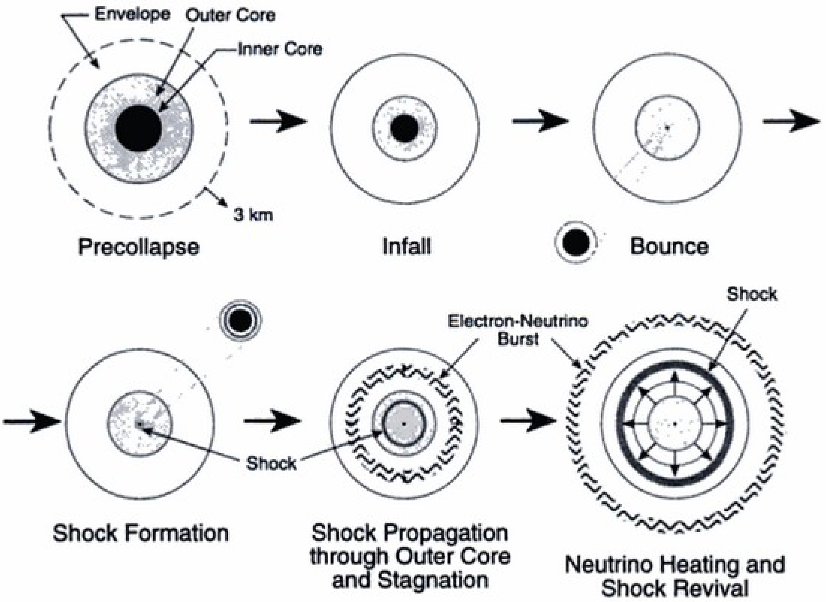
$$\kappa_a = \frac{\sigma_a}{m_n} = \kappa_o \frac{E_\nu^2}{\mathrm{MeV}^2} = 6.7\cdot 10^{-20}\frac{E_\nu^2}{\mathrm{MeV}^2}\ \mathrm{cm\ g}^{-1}.$$

Neutrino energy deposition can't cause an explosion, since the Eddington limit

$$L_{\nu,Edd} = \frac{4\pi cGM}{\kappa_a} = 1.0\cdot 10^{57}\frac{\mathrm{MeV}^2}{E_\nu^2}\ \mathrm{erg\ s}^{-1} \sim 10^{54}\ \mathrm{erg\ s}^{-1},$$

assuming $< E_\nu > \sim 30$ MeV, is about 100 times larger than the expected L_ν.

Neutrino Supernova Mechanism



Neutrino Heating and Cooling

The neutrinos emitted from the newly-formed neutron star will be thermal with temperature T_ν; the average mean square energy is

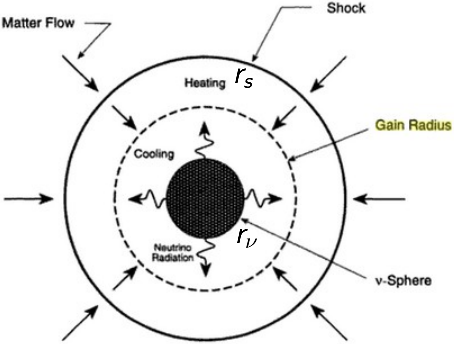
$$< E_\nu^2 >= \frac{\int_0^\infty f_\nu E_\nu^3 E_\nu^2 dE_\nu}{\int_0^\infty f_\nu E_\nu^3 dE_\nu} = \frac{F_5(0)}{F_3(0)} T_\nu^2 = \frac{310\pi^2}{147} T_\nu^2 = 20.8 T_\nu^2$$

where F's are Fermi integrals. ν_e's and $\bar{\nu}_e$'s are emitted roughly equally.

The dominant coupling is through nucleon absorptions of, and thermal radiation through, both ν_e's and $\bar{\nu}_e$'s. The net heating rate per gram is

$$\dot{q} = \frac{7ac}{16}\kappa_o \frac{F_5(0)}{F_3(0)} T_\nu^6 \left[\frac{f}{4}\left(\frac{r_\nu}{r}\right)^2 - \left(\frac{T}{T_\nu}\right)^6 \right].$$

We assumed $\mu_e/T \sim 0$. The 7/16 is from Fermi statistics; $a = \pi^2/[15(\hbar c)^3]$. The heating (cooling) rate is proportional to T_ν^6 (T^6): T^4 from the thermal distribution and T^2 from the energy dependence of the cross section. f is a geometrical factor which is 4 in the opaque limit ($r = r_\nu$) and 1 in the free-streaming limit ($r \to \infty$).



Neutrino Re-Heating

Where T is small, there is net heating. The maximum temperature is reached when heating and cooling balance, and then, using $r_7 = r/10^7$ km,

$$T = T_\nu \left(\frac{\sqrt{f}\,r_\nu}{2r}\right)^{1/3} \simeq 0.5\frac{T_\nu}{r_7^{1/3}}.$$

Ordinarily, one expects that as the protoneutron star loses neutrinos, that the average neutrino energy and T_ν should decrease. However, because the loss of neutrinos leads to a loss of electrons as well, because of beta equilibrium, this is accompanied by a loss of pressure, leading to compressional heating. Therefore, while leptons are being lost near the neutrinosphere, T_ν actually rises, which is helpful.

There is also an overall decrease in density at a given radius, as the infalling matter thins with time, so the optical depth decreases, revealing higher temperature regions nearer the core, which also leads to an increasing T_ν with time.

Early after shock formation, perhaps 20 ms after bounce, the matter accreted through the shock has a density $\rho \simeq 10^{10}$ g cm^{-3} and is electron-rich. The electron capture timescale is short, however, $\tau_{cap} \sim 5(2\rho_{10}Y_e)^{-5/3}$ ms. There is thus net electron and pressure loss, so the mantle (above the core, behind the the shock) sinks and accretes onto the core.

Conditions for Shock Success

Self-similar arguments show that the pre- and post-shock densities vary as

$$\rho_{pre} \propto \rho_{post} \propto r^{-3/2} t^{1-3\gamma/2} \propto r^{-3/2} t^{-1}$$

with $\gamma \sim 4/3$. As time goes on, τ_{cap} decreases. $\tau_{cap} \simeq t$ when $\rho_{10} \simeq 0.1$, halting electron captures.

Rarefaction of matter also leads it to become radiation-dominated, which occurs when, using $T = T_{max}$,

$$\rho = \rho_{rd} = 4 \cdot 10^9 \left(\frac{T}{2.5 \text{ MeV}}\right)^3 \text{ g cm}^{-3} \simeq 4 \cdot 10^9 \left(\frac{T_\nu}{5 \text{ MeV}}\right)^3 \frac{1}{r_7} \text{ g cm}^{-3}.$$

As long as matter is matter pressure-dominated, the specific internal energy is kept approximately constant at a given radius, since $T \simeq T_{max}$. The gravitational specific energy

$$-E_g \simeq = \frac{GM}{r} \simeq 2 \cdot 10^{19} \frac{M}{1.5 M_\odot} \frac{1}{r_7} \text{ erg g}^{-1},$$

is independent of ρ. However, the specific internal energy of radiation-dominated matter $E_{rd} \propto T^4/\rho$ increases with time.

A critical density is reached when $E_{rd} = |E_g|$. Using $T = T_{max}$, this is

$$\rho_{crit} = 7.4 \cdot 10^8 \left(\frac{T}{5 \text{ MeV}}\right)^4 r_7^{-1/3} \text{ g cm}^{-3}.$$

Time Is On the Supernova's Side

In a similar way, one can show that the radiation pressure will exceed the **ram pressure**, the pressure of matter ahead of the shock, $\rho_{pre} v_{pre}^2$, at the same density.

Once matter is radiation-dominated, it no longer needs to be pushed to escape: it's total energy is neutral.

A crucial assumption in the above is that $T \simeq Tmax$. This assumption depends on the heating timescale, $E_g/\dot{q}$.

$$\tau_H \simeq \frac{E_g}{\dot{q}} \simeq \frac{10^{54}\ \mathrm{erg\ s}^{-1}}{L_\nu} \frac{Mr_7}{1.5 M_\odot} \left(\frac{5\ \mathrm{MeV}}{T_\nu}\right)^2\ \mathrm{ms} \simeq 10 - 100\ \mathrm{ms}.$$

The neutrino luminosity is $L_\nu = (7\pi/16) a c r_\nu^2 T_\nu^4$.

Thus, while heating is not instantaneous, it is faster than the timescales for increases in T_ν, decreases in pre- and post-shock densities, electron capture turn-off, ram pressure decrease, specific energy increases, etc.

Everything depends on the neutrino flux and temperature.

Competition Between Accretion and Heating

The bounce shock stalls within about 20 ms of its creation at a distance of $r_s \sim 100 - 200$ km. Whether it is pushed back onto the core or successfully propagates outwards depends on the rate of mass accretion and neutrino heating of the matter behind the shock. Matter in front of the shock has too low density to absorb much energy from neutrinos.

The dynamical timescale is $\tau_d = r_s/v_s$ where v_s is the velocity of matter behind the shock. If τ_d is smaller than for significant changes in $\dot{M}$, a quasi steady-state is achieved. Once again we examine the Eulerian equations of hydrodynamics:

$$\rho\frac{dv}{dt} = \frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial r} = -\frac{Gm\rho}{r^2} - \frac{\partial P}{\partial r},$$

$$\frac{dm}{dt} = -\dot{M} = \frac{\partial m}{\partial t} + 4\pi r^2\rho v,$$

$$T\frac{ds}{dt} + \sum_i \mu_i\frac{dY_i}{dt} = T\left[\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial r}\right] + \sum_i \mu_i\left[\frac{\partial Y_i}{\partial t} + v\frac{\partial Y_i}{\partial r}\right] = \dot{q}$$

$$= \frac{\partial \varepsilon}{\partial t} + P\frac{\partial(1/\rho)}{\partial t} + v\left[\frac{\partial \varepsilon}{\partial r} + P\frac{\partial(1/\rho)}{\partial r}\right] = \dot{q},$$

where $i = n, p, e$. In quasi steady-state, all the time derivatives $\partial/\partial t$ can be set to zero. Also, $\dot{Y}_e = \dot{Y}_p = -\dot{Y}_n$, and $\dot{M}$ is a constant.

For definiteness, we approximate the dilution factor f in $\dot{q}$ as

$$D = \frac{f}{4}\frac{r_\nu^2}{r^2} = \frac{1}{2}\left[1+\frac{r_\nu^2}{r^2}\right]\left[1-\sqrt{1-\frac{r_\nu^2}{r^2}}\right].$$

The rate of change of the particle fractions follow closely from the form of $\dot{q}$:

$$\dot{q} = \frac{7ac}{16}\kappa_o\frac{F_5(0)}{F_3(0)}T_\nu^6\left[D-\frac{T^6}{T_\nu^6}\left(\frac{F_5(\eta_e)}{F_5(0)}Y_e+\frac{F_5(-\eta_e)}{F_5(0)}(1-Y_e)\right)\right],$$

$$v\frac{\partial Y_e}{\partial r} = \frac{7ac}{16}\kappa_o\frac{F_4(0)}{F_2(0)}T_\nu^5\left[D(1-2Y_e)-\frac{T^5}{T_\nu^5}\left(\frac{F_4(\eta_e)}{F_4(0)}Y_e-\frac{F_4(-\eta_e)}{F_4(0)}(1-Y_e)\right)\right].$$

We assumed that $\mu_{\nu_e} = -\mu_{\bar{\nu}_e} = 0$, so

$$L_{\nu_e} = L_{\bar{\nu}_e} = \frac{7ac}{16}\pi r_\nu^2 T_\nu^4,$$

and $\eta_e = \mu_e/T$. We also assumed the absence of heavy nuclei in the post-shock region. The terms proportional to D account for ν_e and $\bar{\nu}_e$ captures on nucleons, and the others account for the inverse reactions of electron and positron captures, respectively, on nondegenerate free nucleons.

We neglected energy transfers from ν-lepton and ν-nucleon scattering, which is an order 10% effect.

Steady-State Solution

Assuming steady-state, $\partial/\partial t = 0$ and $\partial/\partial r = d/dr$, and a constant mass $m(r) = M$, we have

$$\frac{1}{v}\frac{dv}{dr} = -\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}, \qquad v\frac{dv}{dr} = -\frac{GM}{r^2} - \frac{1}{\rho}\frac{dP}{dr}, \qquad \rho\frac{d\varepsilon}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} + \dot{q}.$$

ε and P are functions of ρ, Y_e and T. We can eliminate dv/dr.

$$\frac{d\rho}{dr}(c_s^2 - v^2) = \frac{2\rho}{r}\left(v^2 - \frac{GM}{r}\right) - \frac{A}{C_V}\frac{\dot{q}}{v} - B\frac{dY_e}{dr},$$

$$C_V\frac{dT}{dr}(c_s^2 - v^2) = \frac{2\rho C}{r}\left(v^2 - \frac{GM}{r}\right) + \left(c_T^2 - v^2\right)\frac{\dot{q}}{v} - \frac{dY_e}{dr}\left[\left(c_T^2 - v^2\right)\left(\frac{\partial\varepsilon}{\partial Y_e}\right)_{\rho,T} + C\left(\frac{\partial P}{\partial Y_e}\right)_{\rho,T}\right],$$

We used

$$A = \left(\frac{\partial P}{\partial T}\right)_{\rho,Y_e}, \quad B = \left(\frac{\partial P}{\partial Y_e}\right)_{\rho,T} - \frac{A}{C_V}\left(\frac{\partial\varepsilon}{\partial Y_e}\right)_{\rho,T}, \quad C = \frac{P}{\rho^2} - \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{T,Y_e},$$

$$C_V = \left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho,Y_e}, \qquad c_T^2 = \left(\frac{\partial P}{\partial\rho}\right)_{T,Y_e}, \qquad c_s^2 = \left(\frac{\partial P}{\partial\rho}\right)_{s,Y_e} = c_T^2 + AC.$$

Steady-State

As in the self-similar collapse flow, there is a critical, or sonic, point where $|v| = c_s$.

The inner boundary condition is set at the neutrinosphere, $r = r_\nu$, $T = T_\nu$, $L = L_\nu = \pi r_\nu^2 (7ac/16) T_\nu^4$.

The outer boundary condition is set at the shock front, $r = r_s$. Subscript s (f) refers to behind (in front of) the shock. The Rankine-Hugoniot shock jump conditions, using $p_f << p_s$, are

$$\rho_s v_s^2 + P_s = \rho_f v_f^2, \qquad \frac{v_s^2}{2} + \varepsilon_s + \frac{P_s}{\rho_s} = \frac{v_f^2}{2}, \qquad v_f = -\sqrt{\frac{2GM}{r_s}}.$$

The last follows from self-similar results.

We take $\dot{M}$ and m to be constant, so that $v = -\dot{M}/(4\pi r^2 \rho)$ everywhere. Thus

$$\rho_f = \frac{\dot{M}}{4\pi r_s^2}\sqrt{\frac{r_s}{2GM}}, \qquad \rho_s v_s = \rho_f v_f = -\frac{\dot{M}}{4\pi r_s^2}.$$

The distance between r_s and r_ν is established by the condition

$$\int_{r_\nu}^{r_s} \kappa_{\nu_e} \rho dr = 2/3,$$

where $\kappa_{\nu_e} = \kappa_o [F_4(0)/F_2(0)](T_\nu/\mathrm{MeV})^2$ is the opacity of $\nu_e + n \rightarrow p + e^-$.

Steady-State Isothermal Flows

Some insight is gained by considering an isothermal steady-state flow. The steady-state equations

$$v\frac{dv}{dr} = -\frac{GM}{r^2} - \frac{1}{\rho}\frac{dP}{dr}, \qquad \frac{1}{v}\frac{dv}{dr} = -\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}$$

can be made dimensionless by using $x = rc_T^2/(2GM)$ and the Mach number $\mathcal{M} = v/c_T$, where M and $c_T^2 = (\partial P/\partial\rho)_T$ are assumed constant:

$$\left(\mathcal{M} - \frac{1}{\mathcal{M}}\right)\frac{d\mathcal{M}}{dx} = \frac{2}{x} - \frac{1}{2x^2}.$$

One begins the integration at x_ν and $\mathcal{M}_\nu$.

The Bondi solution goes through the sonic point $\mathcal{M}_{sp} = -1$ when $x_{sp} = 0.25$ and $\rho = \rho_{sp} = -\dot{M}c_T^3/(\pi G^2 M^2)$. At other points, the Bondi flow satisfies

$$\mathcal{M}_{Bondi}^2 - \ln\mathcal{M}_{Bondi}^2 = x^{-1} - 3 + 4\ln 4x, \qquad \rho = \rho_{sp}(-16\mathcal{M}_{Bondi}x^2)^{-1}.$$

The Bondi solution separates flows that are always subsonic from those that become supersonic.

Shocks are possible when the Rankine-Hugoniot conditions are satisfied:

$$\rho^-\mathcal{M}^- = \rho^+\mathcal{M}^+, \qquad \rho^-(\mathcal{M}^-)^2 + \rho^- = \rho^+(\mathcal{M}^+)^2 + \rho^+$$

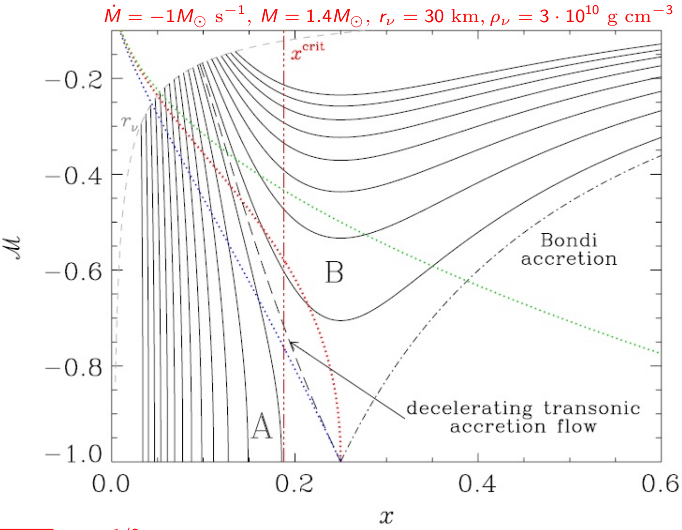
where $+(-)$ refer to up-(down-)stream of the shock. This gives $\mathcal{M}^+\mathcal{M}^- = 1$.

Dotted lines show downstream conditions at the shock where the upstream is Bondi flow (blue) or free-fall (red/green). Dashed lines show the Bondi flow.

Solid lines show downstream flows for fixed $\rho_\nu, r_\nu, \dot{M}$ and M as c_T is increased from left to right.

Where solid lines cross dotted lines, shocks are possible in the flow. Implies $c_T \leq c_T^{crit}, x^{crit} \leq \frac{1}{4}$ for shocks to exist.

When upstream is free-fall, $\mathcal{M}^+ = -x^{-1/2}$ and the solution of the Rankine-Hugoniot relations gives (red dots)



$$\mathcal{M}^- = \mathcal{M}_{ff} = \frac{\sqrt{x^{-1} - 4} - x^{-1/2}}{2}.$$ In this case, $x_{ff}^{crit} = \frac{3}{16}$ and $\mathcal{M}_{ff}^{crit} = -3^{-1/2}$.

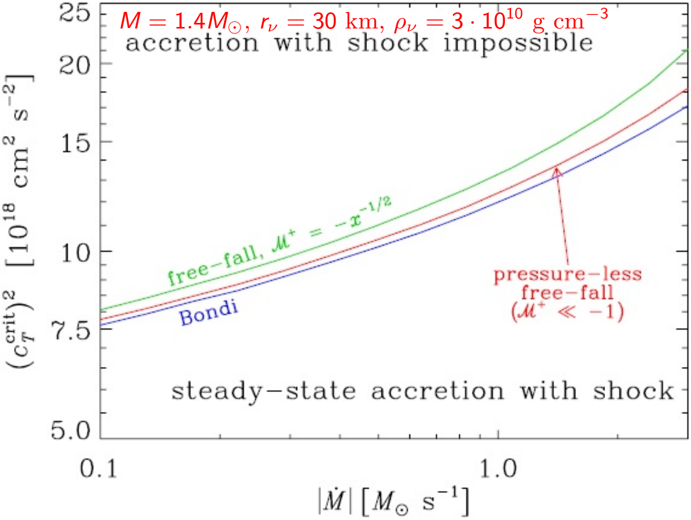
The parameters $\rho_\nu, r_\nu, \dot{M}$ and M determine c_T^{crit}.

$$\mathcal{M}(x_\nu)\sqrt{x_\nu} = \frac{\dot{M}}{4\pi\rho_\nu r_\nu^2}\sqrt{\frac{r_\nu}{2GM}}, \qquad c_T^{crit} = \sqrt{\frac{2GM}{r_\nu}x_\nu}.$$

The region with shocks corresponds to conditions resulting in stalled supernova shocks.

However, changing conditions can lead to the expulsion of the shock and a successful explosion.

In this case, increasing c_T^{crit}, i.e., increasing T, or lowering $\dot{M}$ can increase the chances of a successful supernova shock.



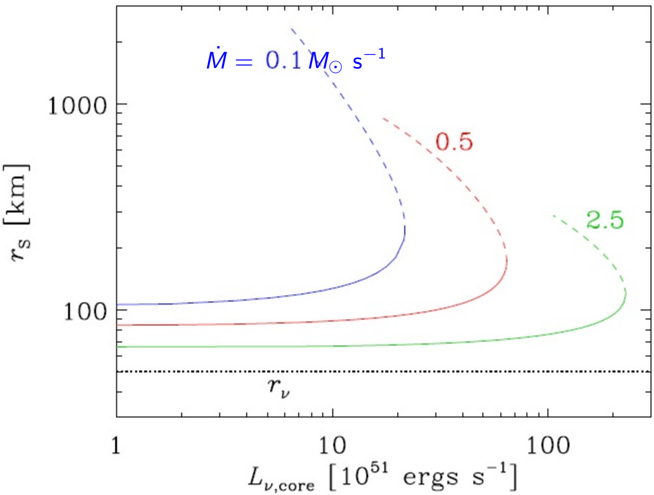
The Full Solution: Shock Radius

In a full steady-state solution, the neutrino luminosity and mass accretion rate determine the location of the shock, in concert with the Rankine-Hugoniot shock jump conditions.

For L_ν below the maximum, there are two solutions for the shock radius.

Only the lower one is relevant in the supernova context.

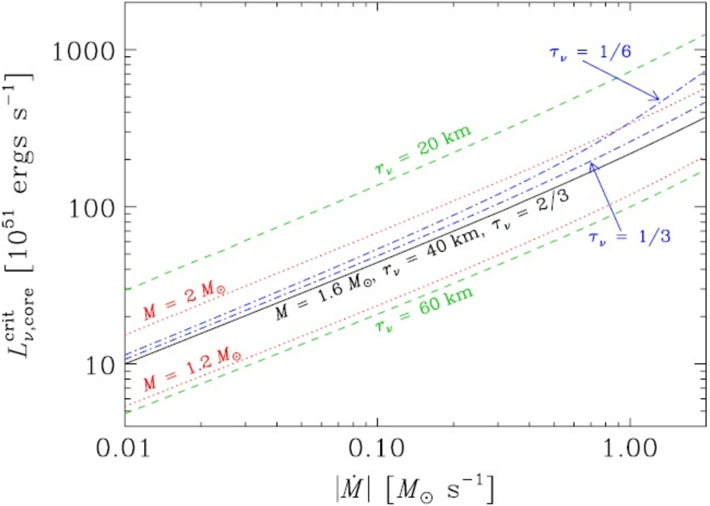
$M = 1.4 M_\odot,\ r_\nu = 50$ km



The Full Solution: Critical Luminosity

Just as observed in the isothermal steady-state case, there is a maximum neutrino luminosity for a given mass accretion rate that will support a shock.

The critical luminosity depends on assumptions for r_ν, M and the optical depth between r_ν and r_s.



An Analytic Solution

Studies show that a number of simplifying approximations can be made.

- The EOS satisfies $h = 4P/\rho$ with no dependence on Y_e, so $dY_e/dr = 0$.
- The density varies as r^{-3}: $\rho = \rho_\nu(r_\nu/r)^3$.
- The neutrino heating rate $\simeq aL_\nu/r^2$, i.e., $f = 1$ and $a = 155\pi\kappa_o T_\nu^2/294$.
- The neutrino cooling rate is $\simeq br^{-4}$. A parcel falling to r_ν, if no heating is present, will radiate the gravitational potential energy

$$\eta\frac{GM}{r_\nu} = \int_\infty^{r_\nu}\frac{4\pi r^2\rho}{\dot{M}}\frac{b}{r^4}dr = -\frac{\pi\rho_\nu b}{\dot{M}r_\nu} \implies b = \eta\frac{GM|\dot{M}|}{\pi\rho_\nu}$$

  where $\eta \sim 0.4$ is the cooling efficiency.
- The net heating rate is then $\dot{q} = aL_\nu/r^2 - b/r^4$.
- Neglect the terms vdv/dr in the Euler equations and those with v_s^2 in the Rankine-Hugoniot conditions at r_s, so the outer boundary conditions become $h(r_s) = v_f^2/2 = GM/r_s$ and $P(r_s) = \dot{M}v_f/(4\pi r_s^2)$.
- Ignore the subsequent Rankine-Hugoniot inconsistency $\rho_s v_s \neq P(r_s)/v_f$.

The Euler equations become, using $dh = dp/\rho$ or $h = (\varepsilon + p)/\rho$,

$$P(r_\nu) - P(r_s) = -\int_{r_s}^{r_\nu}\frac{GM\rho}{r'^2}dr',$$

$$h(r_\nu) - h(r_s) = \frac{GM}{r_\nu} - \frac{GM}{r_s} + \int_{r_s}^{r_\nu}\frac{4\pi r'^2\rho}{\dot{M}}\dot{q}dr'.$$

Analytic Solution Details

Performing the integrals and applying outer boundary conditions yield

$$A\left(\frac{r_s}{r_\nu}\right)^4 + B\left(\frac{r_s}{r_\nu}\right)^2 + C = 0,$$

$$A = \frac{2\pi\rho_\nu r_\nu a L_\nu}{\dot{M}} + \frac{GM\eta}{r_\nu}, \quad B = \frac{\dot{M} v_f}{\pi\rho_\nu r_\nu^2} - \frac{2\pi\rho_\nu r_\nu a L_\nu}{\dot{M}}, \quad C = -\frac{GM}{r_\nu}(1+\eta).$$

Ignoring the weak r_s dependence in B, solving the quadratic shows there are two critical points.

$$\left(\frac{r_s}{r_\nu}\right)^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

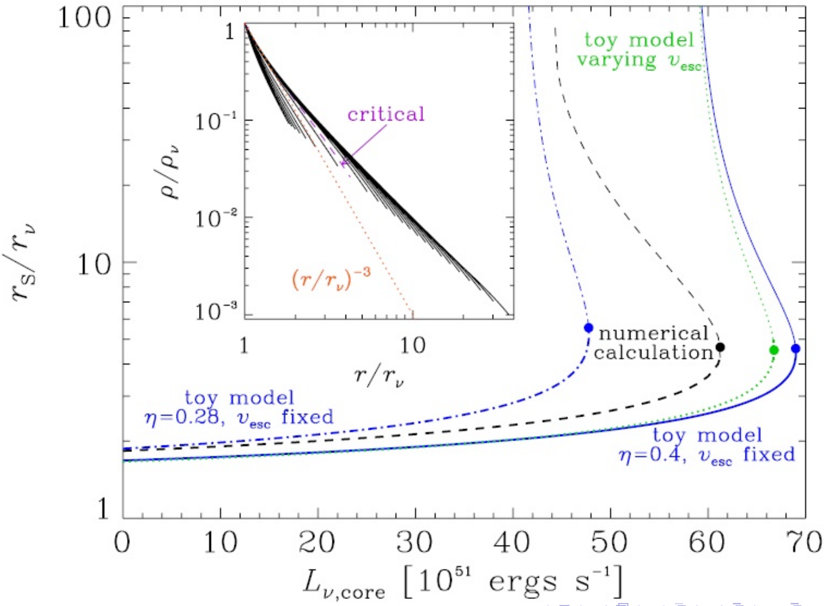
The first occurs when $A = 0$ or $L_\nu = b/(2ar_\nu^2)$, where the$+(-)$ solution is finite(infinite). This gives the minimum L_ν that allows two values of r_s.

The second occurs when $B^2 = 4AC$ and the two solutions merge. This gives the maximum L_ν that allows steady-state solutions. One finds

$$L_\nu^{crit} = \frac{GM|\dot{M}|}{\pi\rho_\nu a r_\nu^2}\left[1 + \eta - \frac{\dot{M} v_f}{2\pi\rho_\nu r_\nu GM} - \sqrt{(1+\eta)\left(1 - \frac{\dot{M} v_f}{\pi\rho_\nu r_\nu GM}\right)}\right].$$

$$L_\nu^{crit} \approx \frac{\eta GM|\dot{M}|}{2\pi\rho_\nu r_\nu^2 a}\left[1 + \frac{\eta}{4}\left(1 + \frac{\dot{M} v_f}{\pi\rho_\nu r_\nu GM\eta}\right)^2 + \cdots\right], \text{valid when } \eta, |\dot{M}| << 1.$$

Analytic vs. Numerical Solution



Analytic vs. Numerical Solution