Darmstadt Lecture 2 – Equation of State

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Pressure - Radius Correlation

Newtonian polytrope: $p = K\rho^{1+1/n}$

$$\Rightarrow R \propto K^{n/(3-n)} M^{(1-n)/(3-n)}$$

Realistic EOS: $n \simeq 1 \Rightarrow R \propto K^{1/2} M^0$

GR phenomenological result:

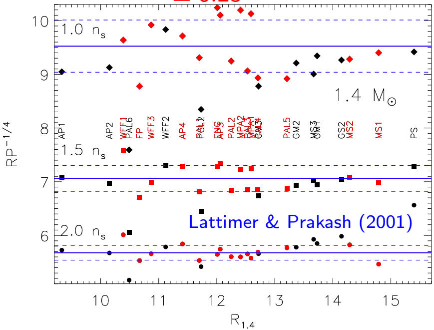
$$R \propto K^{1/4} \propto p_f^{1/4} \rho_f^{-1/2}$$

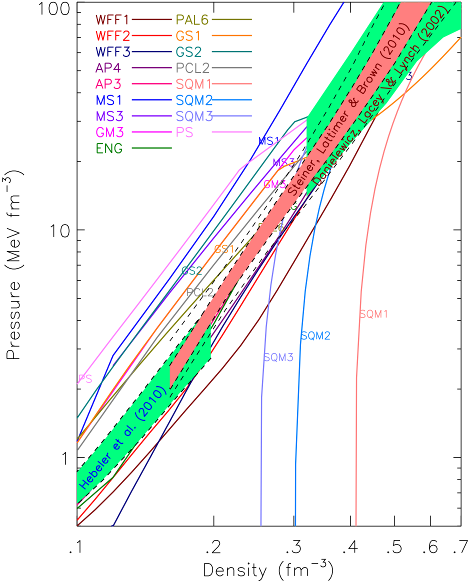
Buchdahl motivation:

$$2R^2 p_*(1-2\beta) = \pi(1-\beta)^2 \Rightarrow$$

$$\left.\frac{d\ln R}{d\ln \rho}\right|_{n,M} = \frac{(1-\beta)(1-2\beta)(\sqrt{p_*}-10\sqrt{p})}{2(1-3\beta+3\beta^2)(\sqrt{p_*}+2\sqrt{p})}$$

$$\simeq 0.23$$



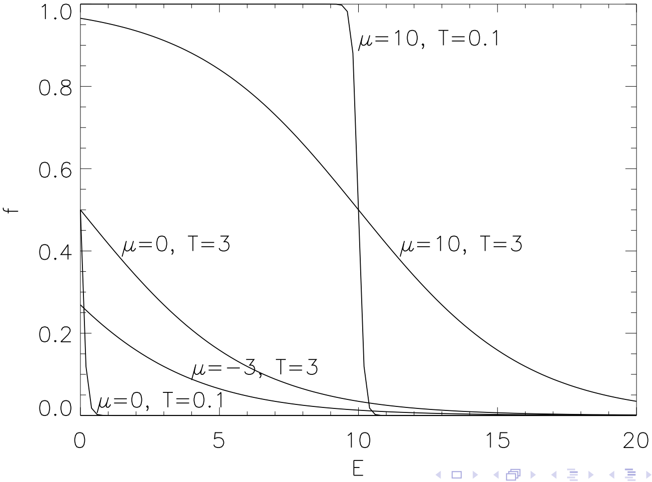


Lattimer & Prakash (2001)

EOS of an Ideal Fermion Gas

Single particle energy $$E^2 = m^2c^4 + p^2c^2$$

Occupation index $$f = \left[1 + \exp\left(\frac{E-\mu}{T}\right)\right]^{-1}$$



Thermodynamics

Number and energy densities

$$n = \frac{g}{h^3}\int_0^\infty f d^3p; \qquad \varepsilon = \frac{g}{h^3}\int_0^\infty E f d^3p; \qquad \mu = \left(\frac{\partial F}{\partial n}\right)_T$$

Landau quasi-particle entropy formula:

$$ns = -\frac{g}{h^3}\int_0^\infty \left[f \ln f + (1-f)\ln(1-f)\right] d^3p$$

Pressure

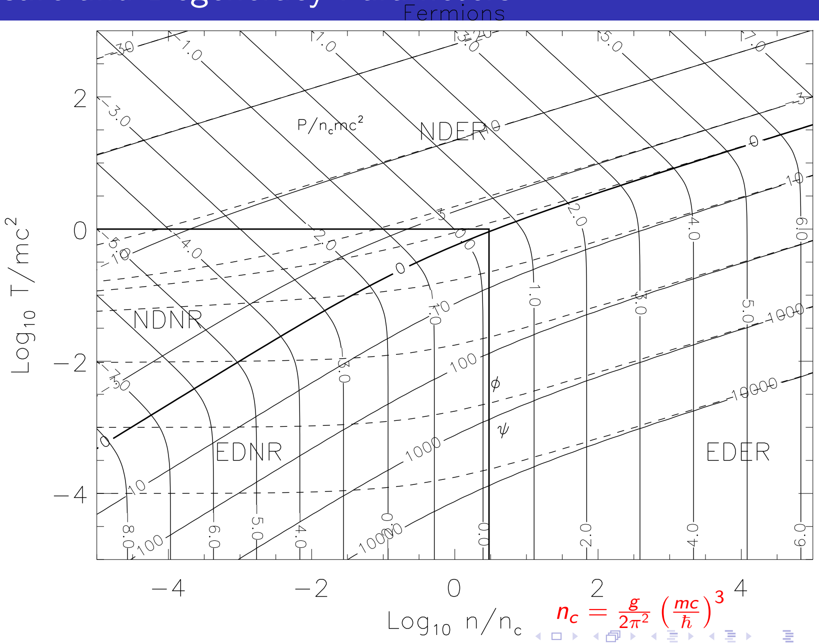
$$P = \frac{g}{3h^3}\int_0^\infty p\frac{\partial E}{\partial p} f d^3p = n^2\left(\frac{\partial (F/n)}{\partial n}\right)_T = \mu n - \varepsilon + nTs = \mu n - F$$

$$n = \left(\frac{\partial P}{\partial \mu}\right)_T; \qquad ns = \left(\frac{\partial P}{\partial T}\right)_\mu$$

Degeneracy parameters

$$\phi = \frac{\mu}{T} = \psi + \frac{mc^2}{T}; \qquad \left(\frac{\partial P}{\partial T}\right)_\phi = n(s+\phi); \qquad \left(\frac{\partial P}{\partial \psi}\right)_T = n(s+\psi)$$

Pressure and Degeneracy Parameters



$$n_c = \frac{g}{2\pi^2}\left(\frac{mc}{\hbar}\right)^3$$

Limits

Non-relativistic $p << mc$

$x = p^2/(2mT)$, $\psi = (\mu - mc^2)/T$

$$n = \frac{g(2mT)^{3/2}}{4\pi^2\hbar^3}\int_0^\infty \frac{x^{1/2}}{1+e^{x-\psi}dx} = \frac{g(2mT)^{3/2}}{4\pi^2\hbar^3}F_{1/2}(\psi)$$

$$\varepsilon = nmc^2 + \frac{gT(2mT)^{3/2}}{4\pi^2\hbar^3}F_{3/2}(\psi)$$

$$P = \frac{2}{3}(\varepsilon - nmc^2), \qquad s = \frac{5F_{3/2}(\psi)}{3F_{1/2}(\psi)} - \psi$$

Relativistic $p >> mc$

$$n = \frac{g}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 F_2(\phi)$$

$$\varepsilon = \frac{gT}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 F_3(\phi)$$

$$P = \frac{1}{3}\varepsilon, \qquad s = \frac{4F_3(\phi)}{3F_2(\phi)} - \phi$$

Limits

$$\left\{\begin{array}{ll} n & = \frac{g}{6\pi^2}\left(\frac{\mu}{\hbar c}\right)^3\left[1+\left(\frac{\pi}{\phi}\right)^2+\cdots\right], \qquad s=\frac{\pi^2}{\phi}+\cdots \\ P & = \frac{1}{3}\varepsilon=\frac{1}{4}n\mu\left[1+\left(\frac{\pi}{\phi}\right)^2+\cdots\right] \end{array}\right\} \text{EDER}$$

$$\left\{\begin{array}{ll} n & = \frac{g}{6\pi^2}\left(\frac{2m\psi T}{\hbar^2}\right)^{3/2}\left[1+\frac{1}{8}\left(\frac{\pi}{\psi}\right)^2+\cdots\right], \qquad s=\frac{\pi^2}{2\psi}+\cdots \\ P & = \frac{2}{3}(\varepsilon-nmc^2)=\frac{2n\psi T}{5}\left[1+\frac{1}{2}\left(\frac{\pi}{\psi}\right)^2+\cdots\right] \end{array}\right\} \text{EDNR}$$

$$\left\{\begin{array}{ll} n & = \frac{g}{\pi^2}\left(\frac{T}{\hbar c}\right)^3 e^{\phi}, \qquad s=4-\phi \\ P & = \frac{1}{3}\varepsilon=nT \end{array}\right\} \text{NDER}$$

$$\left\{\begin{array}{ll} n & = g\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} e^{\psi}; \qquad s=\frac{5}{2}-\psi \\ P & = \frac{2}{3}(\varepsilon-nmc^2)=nT \end{array}\right\} \text{NDNR}$$

Gas With Pairs in Extremely Relativistic Limit

$$\begin{aligned}
\mu &= \mu^+ = -\mu^-, \\
n &= n^+ - n^- = \frac{g}{6\pi^2}\left(\frac{\mu}{\hbar c}\right)^3\left[1 + \left(\frac{\pi T}{\mu}\right)^2\right], \\
P &= P^+ + P^- = \frac{\varepsilon}{3} = \frac{g\mu}{24\pi^2}\left(\frac{\mu}{\hbar c}\right)^3\left[1 + 2\left(\frac{\pi T}{\mu}\right)^2 + \frac{7}{15}\left(\frac{\pi T}{\mu}\right)^4\right], \\
s &= s^+ + s^- = \frac{gT}{6n}\left(\frac{\mu}{\hbar c}\right)^2\left[1 + \frac{7}{15}\left(\frac{\pi T}{\mu}\right)^2\right].
\end{aligned}$$

Cubic Solution : $\mu(n, T)$

$$\begin{aligned}
\mu &= r - q/r, \qquad r = \left[\left(q^3 + t^2\right)^{1/2} + t\right]^{1/3}, \\
t &= \frac{3\pi^2}{g} n(\hbar c)^3, \qquad q = \frac{(\pi T)^2}{3}.
\end{aligned}$$

Interacting Non-Relativistic Fermi Gases

Single particle energy (non-relativistic), $t = n, p$

$$E_t = m_t c^2 + \frac{p^2}{2m_t^*(n_n, n_p)} + V_t(n_n, n_p)$$

$$\frac{\hbar^2}{2m_t^*} = \frac{\delta\varepsilon}{\delta\tau_t}, \qquad V_t = \frac{\delta\varepsilon}{\delta n_t}, \qquad f_t = \left[1 + \exp\left(\frac{E_t - \mu_t}{T}\right)\right]^{-1}$$

Number and kinetic densities

$$n_t = \frac{1}{2\pi^2\hbar^2}\left(\frac{2m_t^* T}{\hbar^2}\right)^{3/2} F_{1/2}(\eta_t); \qquad \tau_t = \frac{1}{2\pi^2\hbar^2}\left(\frac{2m_t^* T}{\hbar^2}\right)^{5/2} F_{3/2}(\eta_t)$$

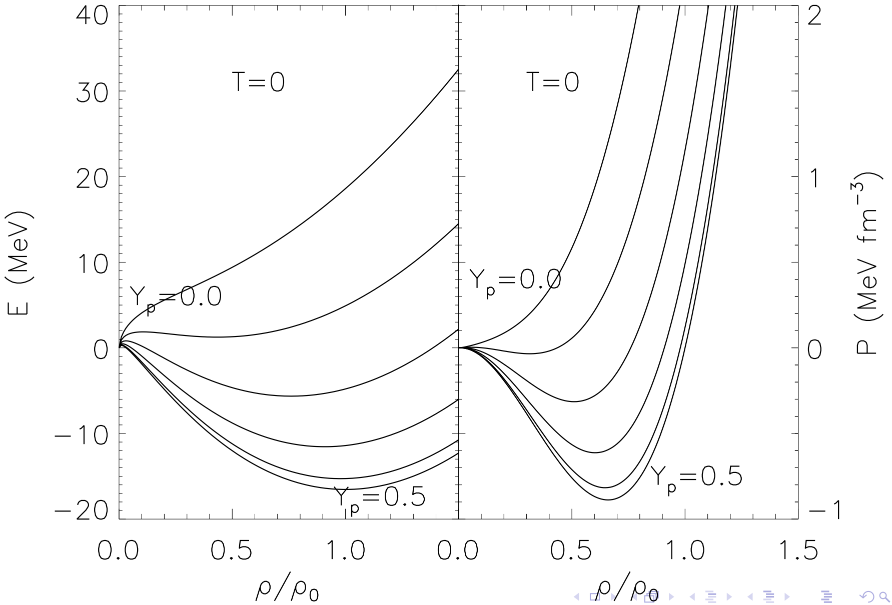
$$\eta_t = \frac{\mu_t - m_t c^2 - V_t}{T}$$

Thermodynamic quantities

$$\varepsilon = \sum_t \frac{\hbar^2}{2m_t^*(n_n, n_p)}\tau_t + U(n_n, n_p); \qquad V_t = \frac{\hbar^2}{2}\sum_s \tau_s \frac{\partial (m_s^*)^{-1}}{\partial n_s} + \frac{\partial U}{\partial n_t}$$

$$P = \sum_t \left[n_t V_t + \frac{\hbar^2 \tau_t}{3m_t^*}\right] - U, \qquad ns = \sum_t \left[\frac{5\hbar^2\tau_t}{6m_t^* T} - n_t\eta_t\right]$$

Bulk Matter Energy and Pressure



Schematic Free Energy Density

$F(n, x, T)$; n: number density; x: proton fraction; T: temperature

$n_s \simeq 0.16 \pm 0.01$ fm^{-3}: nuclear saturation density
$B \simeq 16 \pm 1$ MeV: saturation binding energy
$K_s \simeq 240 \pm 20$ MeV: incompressibility
$S_v \simeq 30 \pm 6$ MeV: bulk symmetry energy
$L \simeq 60 \pm 60$ MeV: symmetry stiffness
$a \simeq 0.065 \pm 0.010$ MeV^{-1}: bulk level density parameter

$K_s' \simeq -200 \pm 200$ MeV: skewness

$K_{sym} \simeq -300 \pm 300$ MeV: symmetry incompressibility

$$F = n\left[-B + \frac{K}{18}\left(1-\frac{n}{n_s}\right)^2 + S_v\frac{n}{n_s}(1-2x)^2 - a\left(\frac{n_s}{n}\right)^{2/3}T^2\right]$$

$$P = n^2\frac{\partial(F/n)}{\partial n} = \frac{n^2}{n_s}\left[\frac{K}{9}\left(\frac{n}{n_s}-1\right) + S_v(1-2x)^2\right] + \frac{2an}{3}\left(\frac{n_s}{n}\right)^{2/3}T^2$$

$$\mu_n = \frac{\partial F}{\partial n} - \frac{x}{n}\frac{\partial F}{\partial x}$$

$$= -B + \frac{K}{18}\left(1-\frac{n}{n_s}\right)\left(1-3\frac{n}{n_s}\right) + 2S_v\frac{n}{n_s}(1-2x) - \frac{a}{3}\left(\frac{n_s}{n}\right)^{2/3}T^2$$

$$\hat{\mu} = -\frac{1}{n}\frac{\partial F}{\partial x} = \mu_n - \mu_p = 4S_v\frac{n}{n_s}(1-2x)$$

$$s = -\frac{1}{n}\frac{\partial F}{\partial T} = 2a\left(\frac{n_s}{n}\right)^{2/3}T; \qquad \varepsilon = F + nTs$$

Phase Coexistence

Negative pressure: matter is unstable to separating into two phases of different densities (and possibly proton fractions). Physically, this represents coexistence of nuclei and vapor. Neglecting finite-size effects, bulk coexistence approximates the EOS at subnuclear densities.

Free Energy Minimization With Two Phases

$$\begin{aligned} F &= \epsilon - nTs = uF_I + (1-u)F_{II}, \\ n &= un_I + (1-u)n_{II}, \\ nY_e &= ux_I n_I + (1-u)x_{II} n_{II}. \end{aligned}$$

$$n_{II} = \frac{n - un_I}{1-u}, \qquad x_{II} = \frac{nY_e - un_I x_I}{n - un_I}$$

$$\begin{aligned} \frac{dF}{dn_I} &= u\frac{\partial F_I}{\partial n_I} + (1-u)\frac{\partial F_{II}}{\partial n_{II}}\left(\frac{-u}{1-u}\right) = u(\mu_{n,I} - \mu_{n,II}) \\ \frac{dF}{dx_I} &= u\frac{\partial F_I}{\partial x_I} + (1-u)\frac{\partial F_{II}}{\partial x_{II}}\left(\frac{-un_I}{n-un_I}\right) = un_I(\hat{\mu}_I - \hat{\mu}_{II}) \\ \frac{dF}{du} &= F_I - F_{II} + (1-u)\left[\frac{\partial F_{II}}{\partial n_{II}}\left(\frac{n_{II}-n_I}{1-u}\right) + \frac{\partial F_{II}}{\partial x_{II}}\left(\frac{-un_I}{n-un_I}\right)\right] \end{aligned}$$

$$\Longrightarrow \mu_{nI} = \mu_{nII}\,, \qquad \mu_{pI} = \mu_{pII}\,, \qquad P_I = P_{II}$$

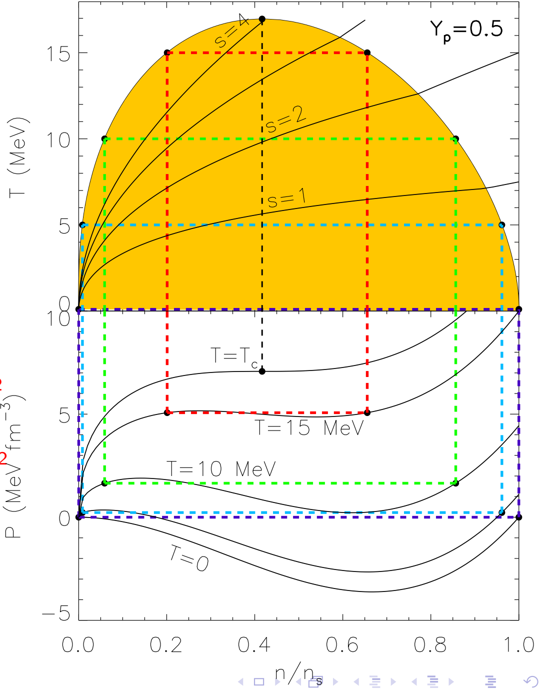
Critical Point
($Y_e = 0.5$)

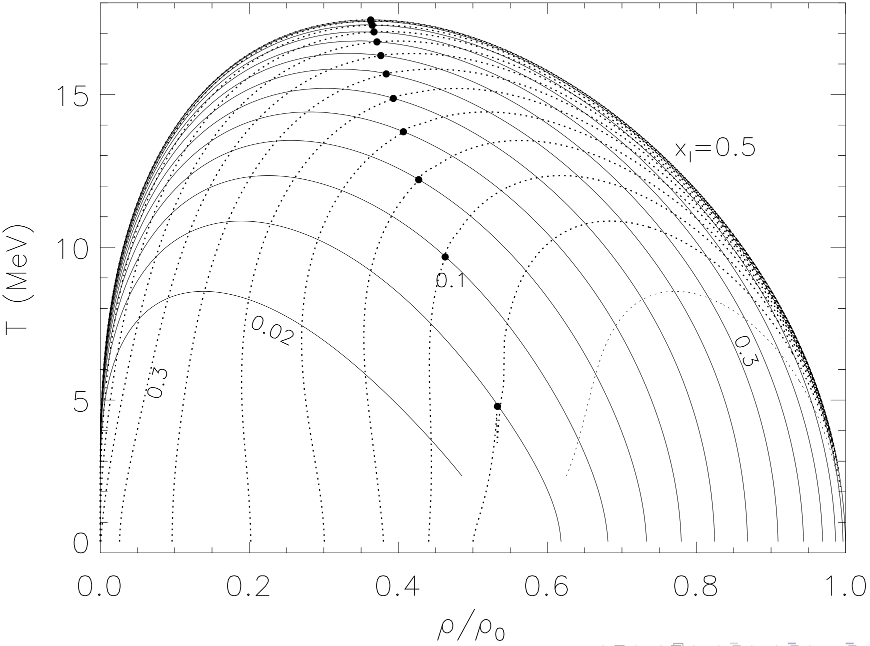
$$\left(\frac{\partial P}{\partial n}\right)_T = \left(\frac{\partial^2 P}{\partial n^2}\right)_T = 0$$

$$n_c = \frac{5}{12} n_s$$

$$T_c = \left(\frac{5}{12}\right)^{1/3} \left(\frac{5K}{32a}\right)^{1/2}$$

$$s_c = \left(\frac{12}{5}\right)^{1/3} \left(\frac{5Ka}{8}\right)^{1/2}$$





15
10
5
0
T (MeV)
0.0
0.2
0.4
0.6
0.8
1.0
ρ/ρ0
x_l=0.5
0.1
0.02
0.3
0.3