Darmstadt Lecture 9 – Global Constraints

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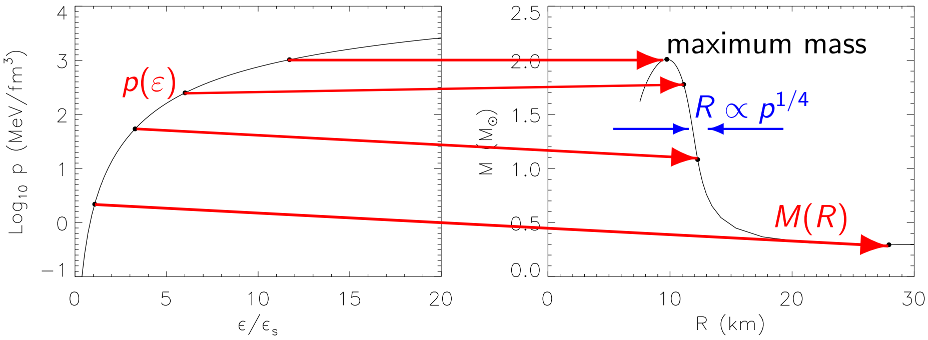
Tolman-Oppenheimer-Volkov equations

$$\frac{dp}{dr} = -\frac{G}{c^2}\frac{(m + 4\pi p r^3/c^2)(\varepsilon + p)}{r(r - 2Gm/c^2)}$$

$$\frac{dm}{dr} = 4\pi\frac{\varepsilon}{c^2}r^2$$



Equation of State ⟵ Observations

Mass-Radius Diagram and Theoretical Constraints

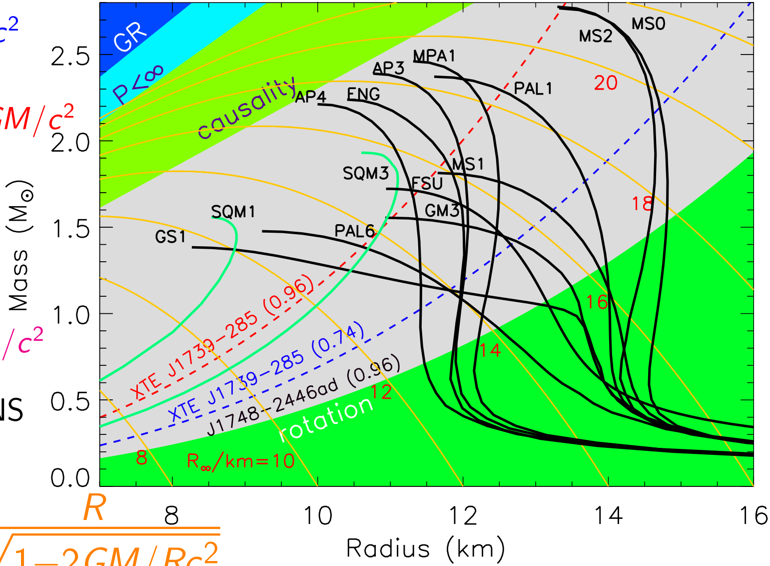
GR:
$R > 2GM/c^2$

$P < \infty$:
$R > (9/4)GM/c^2$

causality:
$R \gtrsim 2.9GM/c^2$

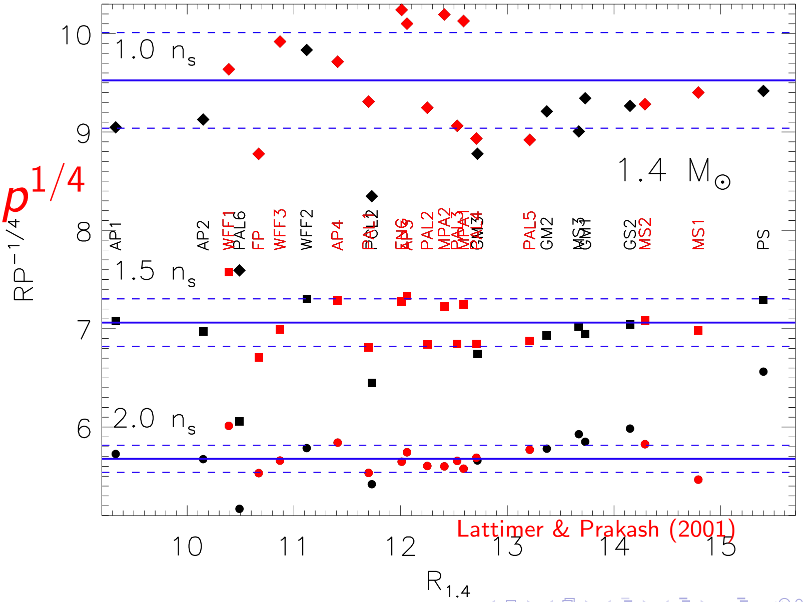
— normal NS
— SQS

— $R_\infty = \frac{R}{\sqrt{1-2GM/Rc^2}}$



The Radius – Pressure Correlation

$R \propto p^{1/4}$



Neutron Star Structure

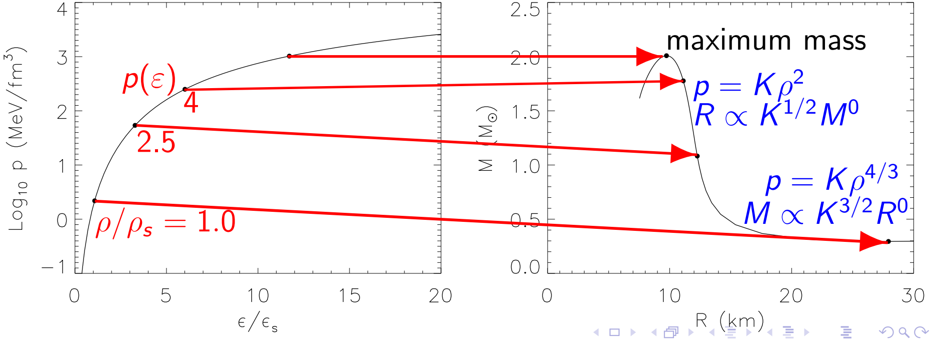
Newtonian Gravity:

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2}; \qquad \frac{dm}{dr} = 4\pi\rho r^2; \qquad \rho c^2 = \varepsilon$$

Newtonian Polytrope:

$$p = K\rho^\gamma; \qquad M \propto K^{1/(2-\gamma)} R^{(4-3\gamma)/(2-\gamma)}$$

$$\rho < \rho_s:\ \gamma \simeq \frac{4}{3}; \qquad \rho > \rho_s:\ \gamma \simeq 2$$



Relativistic Analysis Using Buchdahl's Analytic Solution

$$\varepsilon = \sqrt{pp_*} - 5p,$$

$$R = (1-\beta)\sqrt{\frac{\pi}{2p_*(1-2\beta)}},$$

$$\left.\frac{d\ln R}{d\ln p}\right|_{n,M} = \frac{1}{2}\frac{(1-\beta)(1-2\beta)}{(1-3\beta+3\beta^2)}\frac{(1-10\sqrt{p/p_*})}{(1+2\sqrt{p/p_*})}.$$

$M = 1.4\mathrm{M}_\odot$, $R = 12$ km, $n = 2n_s$, $\varepsilon = 2m_bc^2n_s \simeq 3.98\times10^{-4}$ km^{-2}:

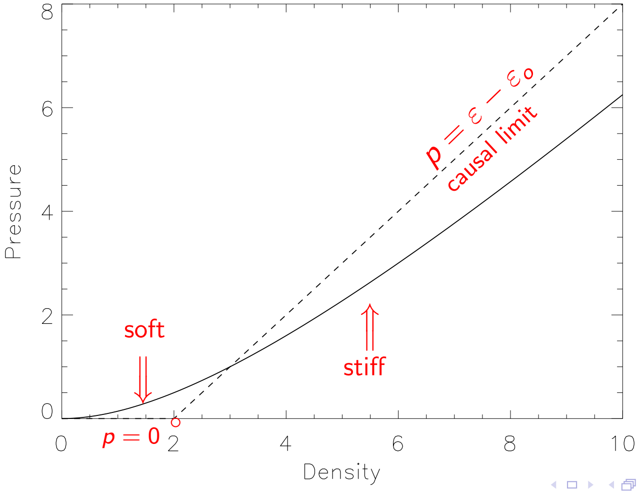
$$\beta = 0.172, \quad p_* = 0.0114\ \mathrm{km}^{-2}, \quad p/p_* = 0.002026.$$

$$\left.\frac{d\ln R}{d\ln p}\right|_{n,M} \simeq 0.239$$

This shows that GR reduces the effective exponent below the Newtonian value of $1/2$.

Extremal Properties of Neutron Stars

The most compact and massive configurations occur when the low-density equation of state is "soft" and the high-density equation of state is "stiff" (Koranda, Stergioulas & Friedman 1997).



ε_0 is the only EOS parameter

The TOV solutions scale with ε_0

$w = \varepsilon/\varepsilon_0$
$y = p/\varepsilon_0 = w - 1$
$x = r\sqrt{G\varepsilon_0}/c^2$
$z = m\sqrt{G^3\varepsilon_0}/c^2$

Extremal Properties of Neutron Stars

The maximum mass configuration is achieved when
$x_R = 0.2404$, $w_c = 3.034$, $y_c = 2.034$, $z_R = 0.08513$.

A useful reference density is the nuclear saturation density (interior density of normal nuclei):
$\rho_s = 2.7 \times 10^{14}$ g cm^{-3}, $n_s = 0.16$ baryons fm^{-3}, $\varepsilon_s = 150$ MeV fm^{-3}

$M_{\rm max} = 4.1\ (\varepsilon_s/\varepsilon_0)^{1/2} M_\odot$ (Rhoades & Ruffini 1974)

$M_{B,\rm max} = 5.41\ (m_B c^2/\mu_o)(\varepsilon_s/\varepsilon_0)^{1/2} M_\odot$

$R_{\rm min} = 2.82\ GM/c^2 = 4.3\ (M/M_\odot)\ \rm km$

$\mu_{b,\rm max} = 2.09$ GeV

$\varepsilon_{c,\rm max} = 3.034\ \varepsilon_0 \simeq 51\ (M_\odot/M_{\rm largest})^2\ \varepsilon_s$

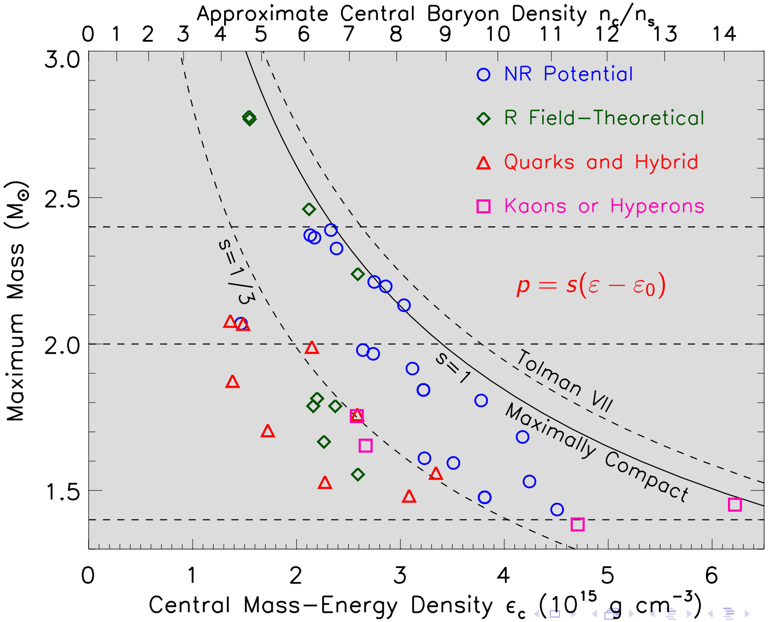
$p_{c,\rm max} = 2.034\ \varepsilon_0 \simeq 34\ (M_\odot/M_{\rm largest})^2\ \varepsilon_s$

$n_{B,\rm max} \simeq 38\ (M_\odot/M_{\rm largest})^2\ n_s$

$\mathrm{BE}_{\rm max} = 0.34\ M$

$P_{\rm min} = 0.74\ (M_\odot/M_{\rm sph})^{1/2}(R_{\rm sph}/10\ \rm km)^{3/2}\ ms = 0.20\ (M_{\rm sph,max}/M_\odot)\ ms$

Maximum Energy Density in Neutron Stars



Causality + GR Limits and the Maximum Mass

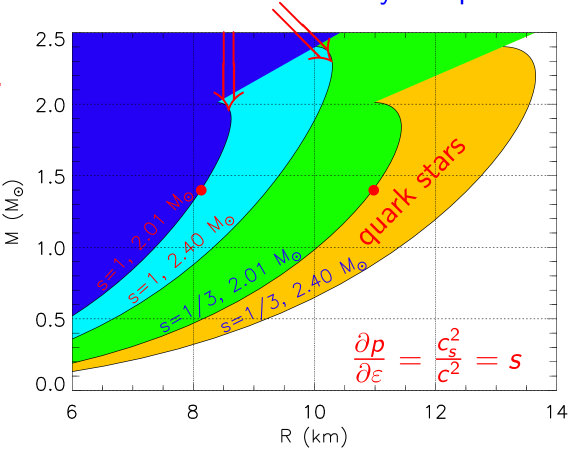
A lower limit to the maximum mass sets a lower limit to the radius for a given mass.

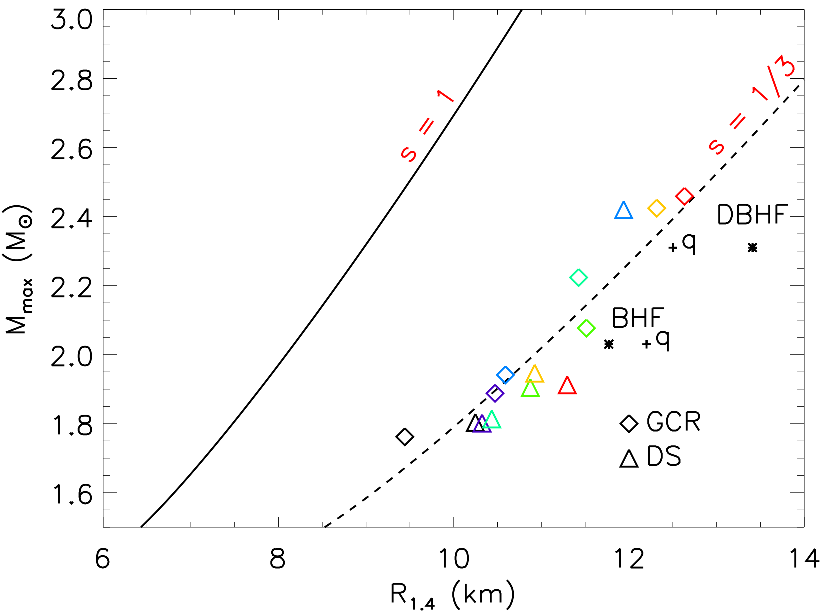
Similarly, a precise (M, R) measurement sets an upper limit to the maximum mass.

$1.4 M_\odot$ stars must have $R > 8.15 M_\odot$.

$1.4 M_\odot$ strange quark matter stars (and likely hybrid quark/hadron stars) must have $R > 11$ km.

$M - R$ curves for maximally compact EOS





3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
6
8
10
12
14
M_{max} ($M_\odot$)
$R_{1.4}$ (km)
s = 1
s = 1/3
DBHF
+q
BHF
+q
◇ GCR
△ DS

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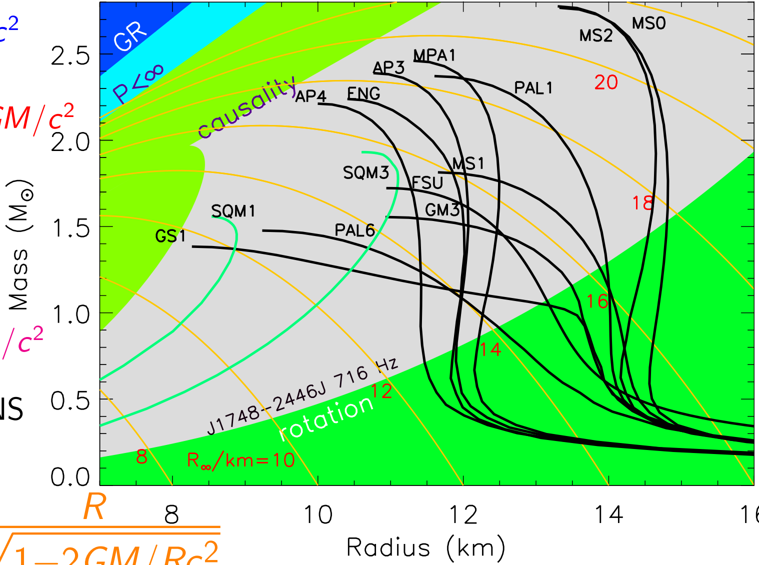

Roche Model for Rotation (Shapiro & Teukolsky 1983)

$$\rho^{-1}\nabla P = \nabla\mu = -\nabla(\Phi_G + \Phi_c)$$

$$\Phi_G \simeq -\frac{GM}{r},\ \Phi_c = -\frac{1}{2}\Omega^2 r^2 \sin^2\theta$$

Bernoulli integral:
$H = \mu + \Phi_G + \Phi_c = -GM/R$
$\mu = \int_0^p \rho^{-1} dp = \mu_n - \mu_{n0}$

Evaluate at equator:

$$\frac{\Omega^2 R_{eq}^3}{2GM} = \frac{R_{eq}}{R} - 1$$

Also true in GR. Mass-shedding limit:
$\Omega_{shed}^2 = GM/R_{eq}^3, \quad R_{eq}/R = \frac{3}{2}$
GR: Cook, Shapiro & Teukolsky (1994):
1.43–1.51

$$\Omega_{shed} = \left(\frac{2}{3}\right)^{3/2}\sqrt{\frac{GM}{R^3}}$$

$$\frac{P_{shed}}{\mathrm{ms}} \simeq 1.00 \left(\frac{R}{10\ \mathrm{km}}\right)^{3/2}\left(\frac{\mathrm{M}_\odot}{M}\right)^{1/2}$$

GR (Haensel et al. 2009): $0.92 \pm 3\%$

Shape: $\dfrac{\Omega^2 R^3 \sin^2\theta}{2GM} = \dfrac{R}{R} - 1$

$$\frac{R}{R(\theta)} = \frac{1}{3} + \frac{2}{3}\cos\left(\frac{1}{3}\cos^{-1}\left[1 - 2\left(\frac{\Omega\sin\theta}{\Omega_{shed}}\right)^2\right]\right)$$

$\Omega \longrightarrow \Omega_{shed}$: $\quad \dfrac{R}{R(\theta)} = \dfrac{\sin(\theta)}{3\sin(\theta/3)}.$