Darmstadt Lecture 13 – Gravitational Waves and Mergers

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Gravitational Waves

In the weak field limit, Einstein's equations can be linearized. For example, one can find coordinates x^α such that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

where $\eta_{\alpha\beta}$ is flat space-time and $|h_{\alpha\beta}| << 1$.

The Einstein field equations involving the Riemann and stress-energy tensors $R_{\alpha\beta}$ and $T_{\alpha\beta}$,

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8\pi T_{\alpha\beta},$$

can also be linearized, if the Lorenz gauge is chosen. One obtains

$$\Box \left(h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta} \right) \equiv \Box \bar{h}_{\alpha\beta} \equiv \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}$$

Gravitational Waves in Space

In the vacuum, $T_{\alpha\beta} = 0$, and we obtain the wave equation for waves travelling at light speed, $\Box\bar{h}_{\alpha\beta} = 0$. It has the solution

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{ik_\mu x^\mu}.$$

$$\Box\bar{h}_{\alpha\beta} = \eta^{\mu\nu} k_\mu k_\nu \bar{h}^{\alpha\beta} = k^\mu k_\mu \bar{h}^{\alpha\beta} = 0,$$

i.e., $\vec{k} = (\omega/c, \mathbf{k})$ is a null vector: $k^\mu k_\mu = \omega - ck = 0$.

The Lorenz gauge requires $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$ leading to the four conditions

$$A^{\alpha\beta} k_\beta = A^{\alpha t} k_t + A^{\alpha i} k_i = 0.$$

Four more conditions arise from the choice of a transverse-tracelless (TT) gauge in which $\eta_{\alpha\beta} A^{\alpha\beta} = 0$, which makes $A^{\alpha\beta}$ traceless and $A^{ti} = 0$. We also find $A^{tt} = 0$ and $A^{t\alpha} = A^{\alpha t} = 0$ from the above equation.

If the wave travels in the z direction, $k_\alpha = (k_t, 0, 0, k_z)$, and we have

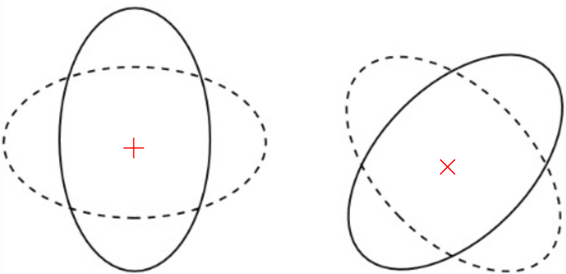
$$A^{\alpha t} k_t + A^{\alpha z} k_z = A^{\alpha z} k_z = 0$$

or $A^{\alpha z} = 0$. The condition $\eta_{\alpha\beta} A^{\alpha\beta} = 0$ means that $A^{xx} + A^{yy} = 0$. So

$$A^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Generation of Gravitational Waves

The constants a and b represent the two polarisations of gravitational waves, which can be separated into tidal distortions oriented 45° with respect to each other and perpendicular to the wave's travel direction.



$\Box\bar{h}^{\alpha\beta} = 2kT^{\alpha\beta}$ is analogous to the EM Poisson equation $\Box\phi = \rho$ which has the solution

$$\phi(t, \mathbf{r}) = \int_V \frac{\rho(t - |\mathbf{r} - \mathbf{x}|/c, \mathbf{x})}{4\pi|\mathbf{r} - \mathbf{x}|} dV.$$

By analogy

$$\bar{h}^{\alpha\beta} = 2k \int_V \frac{T^{\alpha\beta}(t - |\mathbf{r} - \mathbf{x}|/c, \mathbf{x})}{4\pi|\mathbf{r} - \mathbf{x}|} dV \simeq \frac{k}{2\pi r} \int_V T^{\alpha\beta}(t - |\mathbf{r} - \mathbf{x}|/c, \mathbf{x}) dV.$$

In weak gravity, the effect of tides is represented by the equation of geodesic deviation

$$\ddot{\xi}^k \equiv d^2\xi^k/dt^2 = -R^k_{0j0}\xi^j,$$

where ξ^k is the distance between geodesics (i.e., paths of test particle), and R_{k0j0} is the perturbed curvature (i.e., the Riemann tensor in the TT gauge)

$$R_{k0j0} = -\frac{1}{2}\frac{\partial^2 \bar{h}_{jk}}{\partial t^2}.$$

The energy-momentum conservation law $T^{\alpha\beta}{}_{,\beta} = 0$ then gives

$$\bar{h}^{ij} \simeq -\frac{2}{r}\ddot{Q}^{ij}, \qquad Q^{ij} = \int_V \rho\left(x^i x^j - \frac{1}{3}\delta_{ij}r^2\right)dV.$$

Q_{ij} is the quadrupole moment tensor. The stress-energy carried by GWs cannot be localized, but the average $<>$ over many cycles is:

$$T^{\rm GW}_{\mu\nu} = \frac{1}{32\pi}\left\langle\frac{\partial h_{ij}}{\partial x_\mu}\frac{\partial h^{ij}}{\partial x_\nu}\right\rangle.$$

The power carried in GWs is

$$L_{\rm GW} = \frac{dE_{\rm GW}}{dt} = \int T^{\rm GW}_{0j}\hat{n}^j dA = \frac{1}{5}\left\langle\dddot{Q}_{ij}\dddot{Q}^{ij}\right\rangle.$$

One can similarly find the angular momentum GW loss rate:

$$\frac{dJ_{\rm GW}}{dt} = \frac{2}{5}\left\langle\ddot{Q}_{ij}\dddot{Q}^{ij}\right\rangle.$$

Gravitational Waves from a Binary Star

Consider two masses M_1 and M_2 separated by a in a circular orbit in the $x-y$ plane. Kepler's Law gives the angular frequency

$$\Omega = \sqrt{G(M_1+M_2)/a^3}, \qquad \mu = M_1M_2/(M_1+M_2).$$

The quadrupole moment tensor becomes

$$Q_{xx} = \mu a^2\left(\cos^2\Omega t - \frac{1}{3}\right) = \frac{1}{2}\mu a^2\left(\frac{1}{3} + \cos 2\Omega t\right)$$

so that

$$\bar{h}^{xx} = -\bar{h}^{yy} = -\bar{h}^{xy} \simeq -\frac{4G\mu a^2\Omega^2}{c^4 r}\cos 2\Omega t.$$

Note that

- The frequency of the waves is double that of the binary: quadrupolar.
- Amplitude depends on inclination and polarization, averages to 1/2.
- The amplitude decreases as $1/r$ and the power decreases as $h^2 \propto 1/r^2$:

$$h = \frac{1}{2}\frac{4G\mu a^2\Omega^2}{c^4 r} = 2\frac{G(M_1+M_2)}{ac^2}\frac{G\mu}{rc^2} = 2\frac{GM_1}{ac^2}\frac{GM_2}{rc^2}.$$

- A source at the galactic center, $r = 8$ kpc distant, containing two $M_1 = M_2 = 1.4M_\odot$ stars separated by $1R_\odot$, has $h \simeq 2\cdot 10^{-22}$, equivalent to a distortion of 0.03 mm in the distance to α Centauri.

More Binary Relations

More generally, and after much more algebra, one can find relations for eccentric binaries for which $E = -GM_1M_2/(2a)$ and $J = \mu\Omega a^2\sqrt{1-e^2}$:

$$\frac{dE_{\rm GW}}{dt} = -\frac{32G^4}{5c^5}\frac{M_1^2M_2^2(M_1+M_2)}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right),$$

$$\frac{dJ_{\rm GW}}{dt} = -\frac{32G^{7/2}}{5c^5}\frac{M_1^2M_2^2(M_1+M_2)^{1/2}}{a^{7/2}(1-e^2)^2}\left(1+\frac{7}{8}e^2\right),$$

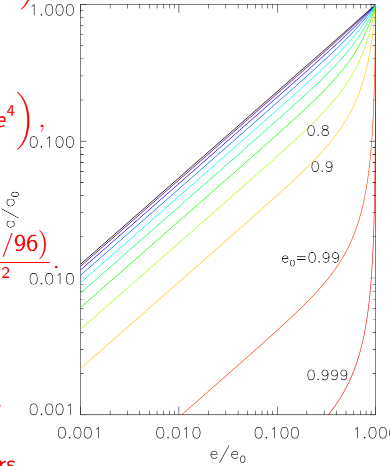
$$\frac{da}{dt} = -\frac{64G^3}{5c^5}\frac{M_1M_2(M_1+M_2)}{a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right),$$

$$\frac{de}{dt} = -\frac{304G^3}{15c^5}\frac{M_1M_2(M_1+M_2)}{a^4(1-e^2)^{5/2}}e\left(1+\frac{121}{304}e^2\right),$$

$$\frac{dP_b}{dt} = -\frac{192\pi(G\Omega)^{5/3}}{5c^5}\frac{M_1M_2\left(1+73e^2/24+37e^4/96\right)}{(M_1+M_2)^{1/3}(1-e^2)^{7/2}}.$$

da/de is exactly integrable so that

$$\frac{a}{a_0} = \left(\frac{e}{e_0}\right)^{12/19}\frac{1-e_0^2}{1-e^2}\left(\frac{1+121e^2/304}{1+121e_0^2/304}\right)^{870/2299}.$$



Eccentric orbits circularize long before a merger occurs unless $1 - e_0 << 1$.

Merger Times

For circular orbits, one can integrate the $\dot{a}$ equation to find

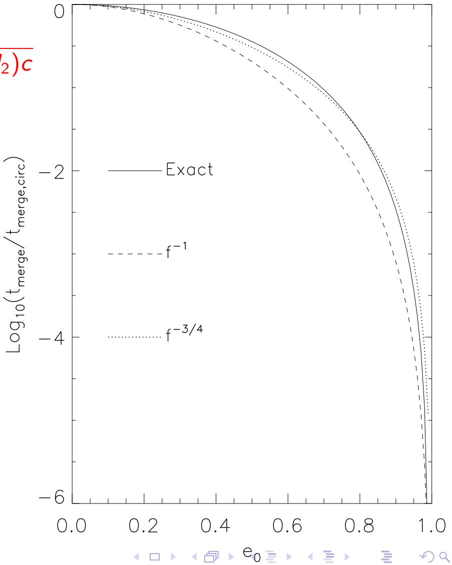
$$\tau_{merge,circ} = \frac{5}{256}\left(\frac{a_0 c^2}{G}\right)^3 \frac{a_0}{M_1 M_2 (M_1 + M_2) c}$$

which is 0.146 Gyr for $M_1 = M_2 = 1 M_\odot$ and $a_0 = R_\odot$.

If $e_0 > 0$, merger times are shorter. Since e decays more rapidly than a, the shortening factor is approximately

$$\frac{1 + (73/24)e_0^2 + (37/96)e_0^4}{(1 - e_0^2)^{7/2}}.$$

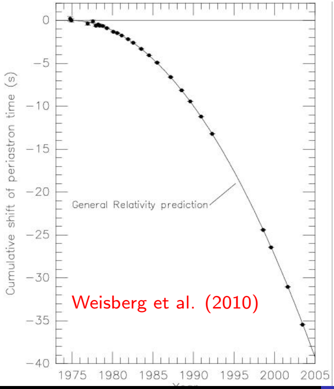
Most of the observed double neutron star binaries will decay in less than 10 Gyr.



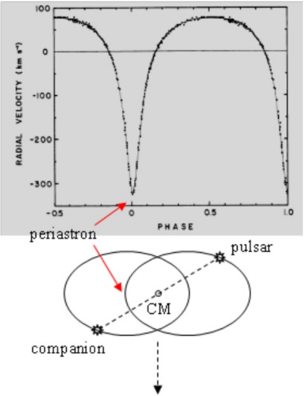
The Binary Pulsar PSR B1913+16

Discovered by Hulse & Taylor in 1974. Over 40 years of timing gives incredibly small uncertainties: $M_1 = 1.4398 \pm 0.0002 M_\odot$, $M_2 = 1.3886 \pm 0.0002 M_\odot$, $a_1 \sin i/c = 2.341782(3)$ s, $e = 0.6171334(5)$, $P_b = 0.322997448911(4)$ d, pulsar spin $f = 16.94053778563(15)$ Hz, $\dot\omega = 4.226598(5)$ d yr^{-1}, $\dot P_b = -2.423(1) \times 10^{-12}$, and relativistic factor

$$\gamma = G^{2/3} \frac{e}{c^2} \left(\frac{P_b}{2\pi}\right)^{1/3} \frac{M_2(M_1 + 2M_2)}{(M_1 + M_2)^{4/3}} = 4.2992(8) \text{ ms}.$$



Weisberg et al. (2010)



A Merger in Gravitational Waves

Write the equations in terms of mass and signal frequency $f = \Omega/\pi$:

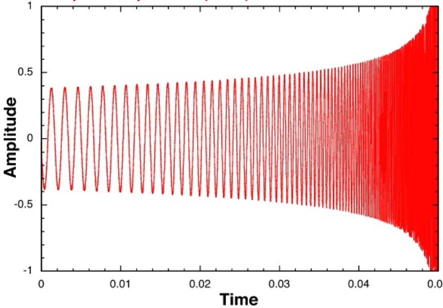
$$h_{+,\times} = \frac{2}{r}\left(\frac{\pi f}{c}\right)^{2/3}\left(\frac{G\mathcal{M}}{c^2}\right)^{5/3}[\cos(2\pi ft), \sin(2\pi ft)].$$

$$\textbf{ChirpMass}: \mathcal{M} = \frac{(M_1 M_2)^{3/5}}{(M_1+M_2)^{1/5}} = \frac{c^3}{G}\left(\frac{5\pi\dot{f}}{96}\right)^{3/5}\left(\frac{1}{\pi f}\right)^{8/5}.$$

$$\frac{\dot{f}}{f} = \frac{3}{2}\frac{\dot{h}}{h} = \frac{96}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}(\pi f)^{8/3},$$

$$\frac{f(t)}{f_0} = \left[1 - \frac{256}{5}(\pi f_0)^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} t\right]^{-3/8};$$

merger occurs at $t = 0$ and $f = f_0$.



These sources are therefore **standard sirens** to determine the **luminosity distance** independently of $\mathcal{M}$ ($r, \mathcal{M}, 1/f, \dot{f}^{-1/2}$ all scale as $(1+z)$):

$$r = \frac{5c}{48\pi^2}\frac{\dot{f}}{hf^3}.$$

Merger Properties

Naively, integrating $dE_{\rm GW}/da$ from a large distance until two objects touch would result in an energy change ($G = C = 1$)

$$\Delta E = -\frac{M_1 M_2}{2(R_1 + R_2)} \simeq -\frac{\mu}{4}.$$

For black holes, $R_i = 2M_i$. For equal masses M the energy release is $M/8$, and the final black hole mass is $M_f = (15/8)M$.
Just before touching, the orbital angular momentum would be

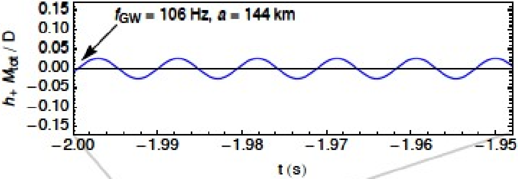
$$J = \mu\sqrt{(M_1 + M_2)a} = \sqrt{2}M_1 M_2 = \sqrt{2}M^2$$

This add to the spin angular momentum of the black hole. With no further GW emission, the final black holes's Kerr parameter becomes

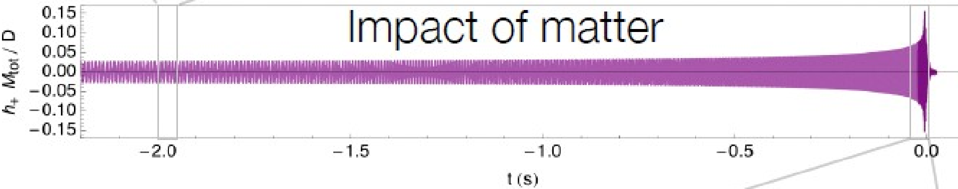
$$\frac{a_f}{M_f} = \frac{J_i + J_f}{M_f^2} = \frac{a_{1i}M_1 + a_{2i}M_2 + \sqrt{2}M^2}{M_f^2} = \left(\frac{8}{15}\right)^2 \left[\frac{a_{1i} + a_{2i}}{M} + \sqrt{2}\right] \simeq 0.40$$

if $a_{1i} = a_{2i} = 0$. If $a_{1i} = a_{2i} = 0.3M$, we obtain $a_f = 0.57M_f$.

Impact of matter



Hard to modify inspiral: transfer of ~10^{46} erg at ~100 Hz modifies phase by 10^{-3} radians (Crust shattering, Tsang et al 1110.0467)

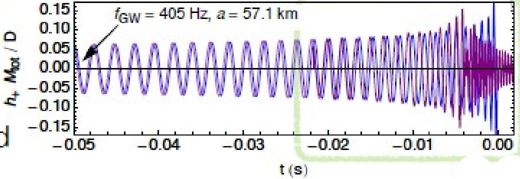


Tidal interactions lead to accumulated phase shift at higher frequencies.

$$\delta\Phi_t = -\frac{117}{256}\frac{(1+q)^4}{q^2}\left(\frac{\pi f_{GW} G\mathcal{M}}{c^3}\right)^{5/3}\bar{\Lambda}$$

For the final coalescence, numerical simulations are required

credit: Jocelyn Read



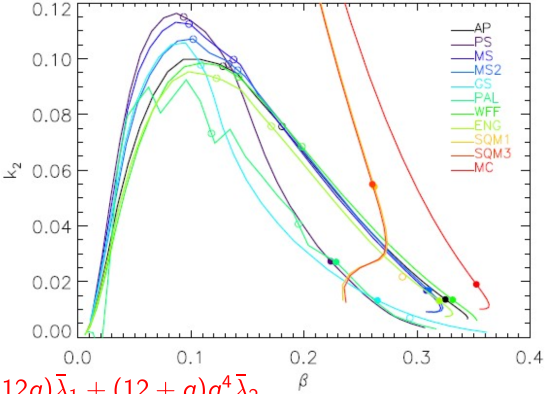
Tidal Deformability

Tidal deformability λ is the ratio between the induced dipole moment Q_{ij} and the external tidal field E_{ij},

k_2 is the dimensionless Love number. It is convenient to work with the dimensionless

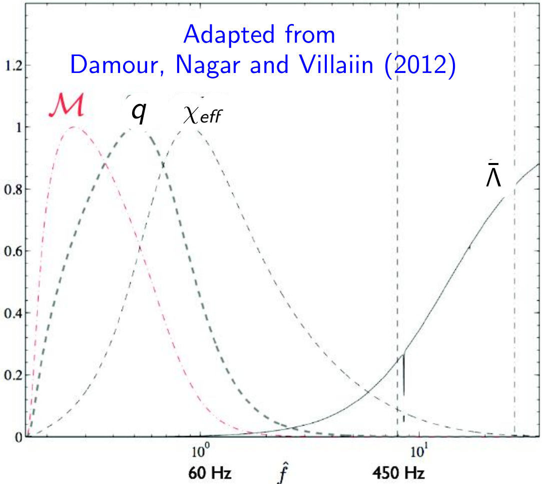
$$\Lambda = \frac{\lambda c^{10}}{G^4 m^5} \equiv \frac{2}{3} k_2 \left(\frac{Rc^2}{Gm}\right)^5$$

For a binary neutron star, the relevant quantity is $(q = m_2/m_1 \leq 1)$



$$\bar{\Lambda} = \frac{16}{13} \frac{(1 + 12q)\bar{\lambda}_1 + (12 + q)q^4 \bar{\lambda}_2}{(1 + q)^5}.$$

When We Know What



In a post-Newtonian expansion, spin effects can be characterized by a single parameter χ_{eff}.

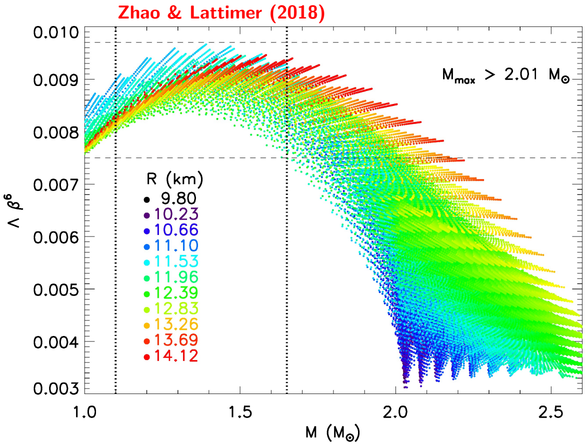
For spins aligned with $\vec{L}$, spin effects act oppositely to tides.

Λ is Highly Correlated With M and R

- $\Lambda = a\beta^{-6}$
  $\beta = GM/Rc^2$

  $a = 0.0086 \pm 0.0011$
  for
  $M = 1.35 \pm 0.25\ M_\odot$

- If $R_1 \simeq R_2 \simeq R_{1.4}$
  it follows that
  $\Lambda_2 \simeq q^{-6}\Lambda_1$.

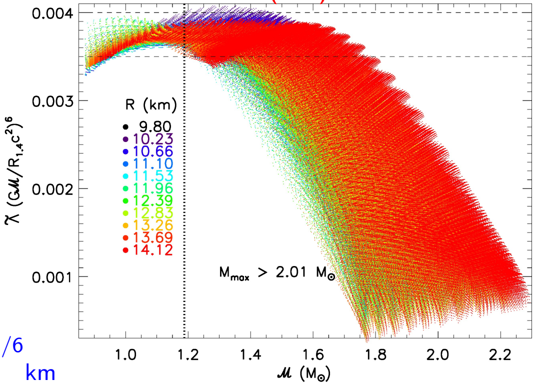


Binary Deformability and the Radius

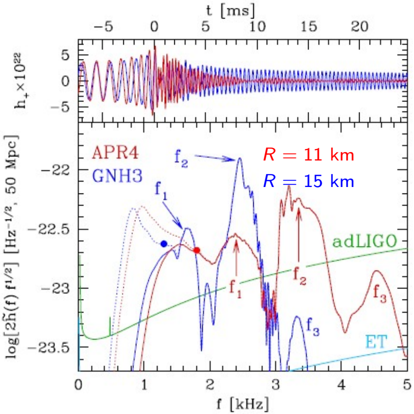
$$\tilde\Lambda=\frac{16}{13}\frac{(1+12q)\Lambda_1+q^4(12+q)\Lambda_2}{(1+q)^5}\simeq\frac{16a}{13}\left(\frac{R_{1.4}c^2}{G\mathcal{M}}\right)^6\frac{q^{8/5}(12-11q+12q^2)}{(1+q)^{26/5}}$$

- $\tilde\Lambda=a'(R_{1.4}c^2/G\mathcal{M})^6$
  $a'=0.0035\pm0.0006$
  for
  $\mathcal{M}=1.2\pm0.2\ M_\odot$
- GW10817:
  $a'=0.00375\pm0.00025$
- $R_{1.4}=$
  $11.5\pm0.3\ \frac{\mathcal{M}}{M_\odot}\left(\frac{\tilde\Lambda}{800}\right)^{1/6}$ km
- GW10817:
  $R_{1.4}=13.4\pm0.1\ \left(\frac{\tilde\Lambda}{800}\right)^{1/6}$ km

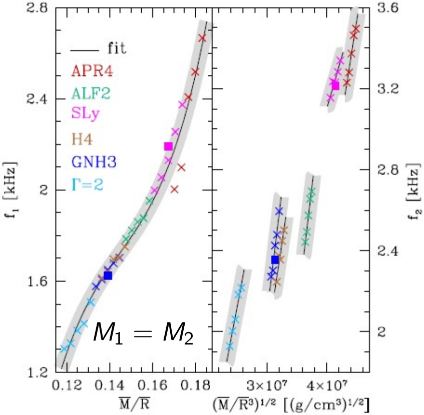
Zhao & Lattimer (2018)



GW Post-Merger Constraints



Takami, Rezzolla and Baiotti (2014)



- Chirp mass $\mathcal{M} = (M_1 M_2)^{3/5}(M_1 + M_2)^{-1/5}$ and tidal deformability $\Lambda \propto k_2(R/M)^5$ measurable during inspiral.
- Frequency peaks are tightly correlated with $\sqrt{M/R^3}$.
- Maximum mass from prompt vs. delayed black hole formation.
- In neutron star-black hole mergers, disc mass depends on a/M_{BH} and on $M_{NS}M_{BH}/R^2$.