Darmstadt Lecture 1 – History and Global Structure

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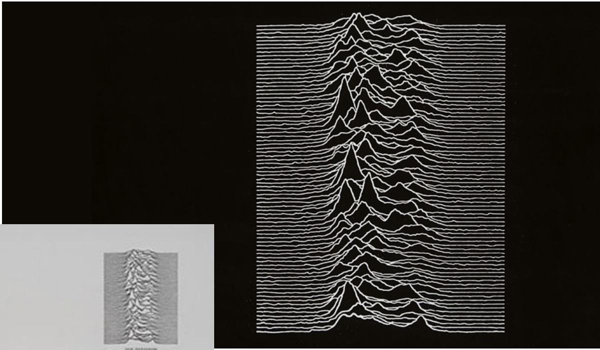
Pulsars: The Early History

1932 Landau suggests the existence of giant nucleus stars.
1932 Chadwick discovers the neutron.
1934 Baade & Zwicky predicts the existence of neutron stars as the end products of supernovae.
1939 Oppenheimer and Volkoff predict the upper mass limit of neutron star.
1964 Hoyle, Narlikar and Wheeler predict neutron stars rapidly rotate.
1964 Prediction that neutron stars have intense magnetic fields.
1966 Colgate and White incorporate neutrinos into supernova hydrodynamics.
1966 Wheeler predicts the Crab nebula is powered by a rotating neutron star.
1967 Pacini makes the first magnetic pulsar model.
1967 C. Schisler discovers a dozen pulsing radio sources, including one in the Crab pulsar, using secret military radar in Alaska.
1967 Hewish et al. discover the pulsar PSR 1919+21, Aug 6.
1968 Discovery of a pulsar in the Crab Nebula which was slowing down, ruling out binary models. Also clinched the connection with core-collapse supernovae.
1968 T. Gold identifies pulsars with rotating magnetized neutron stars.
1968 The term "pulsar" first appears in print, in the *Daily Telegraph*.
1969 Vela pulsar glitches observed; evidence for superfluidity in neutron stars.
1971 Accretion powered X-ray pulsar discovered by Uhuru (*not* the Lt.).
1974 Hewish awarded Nobel Prize (but Jocelyn Bell Burnell was not).

Pulsars: Later Discoveries

1974 Lattimer & Schramm suggest neutron star mergers make the r-process.
1974 First binary pulsar, PSR 1913+16, discovered by Hulse and Taylor.
1979 Taylor et al. observe orbital decay due to gravitational radiation in the PSR 1913+16 system, leading to their Nobel Prize in 1993.
1979 Chart recording of PSR 1919+21 used as album cover for *Unknown Pleasures* by Joy Division (#19/100 greatest British albums ever).
1982 First millisecond pulsar, PSR B1937+21, discovered by Backer et al.
1992 Wolszczan and Frail find first extra-solar planets around PSR B1257+12.
1988 First black widow pulsar, PSR 1957+20, discovered by Fruchter et al.
1992 Duncan & Thompson predict existence of magnetars.
1994 Discovery of PSR J0108-1431 which has the lowest known dispersion measure and is possibly the closest known pulsar.
1998 Li & Paczynski propose neutron star mergers make kilonovae.
2004 Largest burst of energy in our Galaxy since Kepler's SN 1604 is observed from magnetar SGR 1806-20. It was brighter than the full moon in gamma rays and radiated more energy in 0.1 second than the Sun does in 100,000 years.
2004 Hessels et al. discover the fastest (716 Hz) pulsar, PSR J1748-2446ad.
2005 Burgay et al. discover the first binary with two pulsars, PSR J0737-3039.
2013 Measurement of largest mass, 2.01 $M_\odot$, for PSR J0348+0432.
2013 Ransom et al. discover a pulsar in a triple system with 2 white dwarfs.
2017 LVC discovers the first binary neutron star merger, GW170817.



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Stellar Structure in the Newtonian Limit

In the Newtonian limit, the structure equations are

$$\frac{dm}{dr} = 4\pi\rho r^2, \qquad \frac{dp}{dr} = -\frac{Gm\rho}{r^2}$$

There are a number of analytic solutions. $\rho = \rho_c$ =constant yields

$$m = \frac{4\pi}{3}\rho_c r^3, \qquad p = \frac{G\rho_c M}{2R}\left[1 - \left(\frac{r}{R}\right)^2\right].$$

The central pressure is $p_c = G\rho_c M/(2R) = 3GM/(8\pi R^3)$.

Another analytic case is $p = K\rho^2$. Now we find

$$\rho = \rho_c \frac{\sin\sqrt{2\pi G/K}r}{\sqrt{2\pi G/K}r}, \qquad p_c = K\rho_c^2, \qquad M = \frac{4}{\pi}\rho_c R^3, \qquad R = \sqrt{\frac{\pi K}{2G}}.$$

The causality limit, $dp/d\rho = c^2$, is reached when $GM/R = c^2$.

Still another case is $\rho = \rho_c[1 - (r/R)^2]$, leading to

$$p = \frac{4\pi}{15}G\rho_c^2 R^2\left[1 - \frac{5}{2}\left(\frac{r}{R}\right)^2 + 2\left(\frac{r}{R}\right)^4 - \frac{1}{2}\left(\frac{r}{R}\right)^6\right] = \frac{2\pi}{15}G\rho^2 R^2\left(1 + \frac{\rho}{\rho_c}\right),$$

$$M = 8\pi\rho_c R^3/15, \qquad p_c = 4\pi G\rho_c^2 R^2/15.$$

This becomes acausal when $GM/R = 4c^2/5$.

Newtonian Polytropes

In many situations, it can be convenient or a good approximation to assume $p = K\rho^\gamma$ where K and γ are constants for the whole star. Such configurations are called polytropes of index $n = 1/(\gamma - 1)$.

We can determine the gravitational potential energy, where the Newtonian potential is $\phi = -m/r$. Noting that in general

$$d\Omega = -\frac{Gmdm}{r} = -4\pi Gm\rho rdr = 4\pi r^3 p = 3Vdp, \qquad \Omega = -3\int_V pdV,$$

we determine Ω for a polytrope:

$$\begin{aligned}
\Omega &= -\int_V \frac{Gmdm}{r} = -\frac{G}{2}\int_V \frac{d(m^2)}{r} = -\frac{GM^2}{2R} + \frac{G}{2}\int_V m\frac{dp}{\rho} = \\
&= -\frac{GM^2}{2R} + \frac{n+1}{2}G\int_V md\left(\frac{p}{\rho}\right) = -\frac{GM^2}{2R} - \frac{n+1}{2}G\int_V \frac{p}{\rho}dm \\
&= -\frac{GM^2}{2R} - \frac{n+1}{2}G\int_V pdV = -\frac{GM^2}{2R} + \frac{n+1}{6}\Omega \\
&= -\frac{3}{5-n}\frac{GM^2}{R}.
\end{aligned}$$

Polytropes in Nature

Non-degenerate, non-relativistic ionized gas (nuclei + electrons) including radiation pressure, with $\beta = p_{gas}/p_{rad}$ fixed:
Can apply to moderate- to high-mass main-sequence stars, $\gamma = 4/3$.

$$T = \left[\frac{3N_0 k_B}{\mu\beta a}(1-\beta)\right]^{1/3} \rho^{1/3}$$

$$p = \frac{N_0 k_B}{\mu}\rho T + \frac{a}{3}T^4 = \frac{N_0 k_B}{\mu\beta}\left[\frac{3N_0 k_B}{\mu\beta a}(1-\beta)\right]^{1/3} \rho^{4/3},$$

Stars in convective equilibrium, for which the entropy is constant, radiation pressure is negligible, and the gas is non-relativistic.
Can apply to stars forming on the Hayashi track, or to very low-mass main sequence stars, $\gamma = 5/3$.

$$s = \frac{5}{2} - \ln\left[\left(\frac{\hbar^2}{2mk_B T}\right)^{3/2} \rho \frac{N_0 k_B}{\mu}\right] = \text{ constant}$$

$$p = \frac{N_0 k_B}{\mu}\rho T = \frac{\hbar^2 \rho}{2m}\left(\frac{N_0 k_B}{\mu}\right)^{5/3} e^{2s/3-5/3} = K\rho^{5/3}.$$

An isothermal, non-degenerate perfect gas with pairs, radiation and electrostatic interactions negligible. Could be a dense molecular cloud core prior to its collapse, $\gamma = 1$:

$$p = \frac{N_0 k_B}{\mu} \rho T.$$

An incompressible fluid.
A zeroth order approximation to a neutron star, $\gamma \to \infty$.

Non-relativistic degenerate fermions.
Can apply to low-density white dwarfs and cores of evolved stars, $\gamma = 5/3$.

Relativistic degenerate fermions.
Can apply to high-density white dwars, $\gamma = 4/3$.

Cold matter at planetary core densities.
Degenerate non-relativistic electrons plus nuclear Coulomb lattice pressure, but electrons are not uniformly distributed due to Coulomb effects, $\gamma = 10/3$.

Structure of Newtonian Polytropes

Form dimensionless variables $\xi = r/A$ and $\theta = (\rho/\rho_c)^{1/n}$. Substitute these relations into the Newtonian hydrostatic equilibrium equations:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,$$

$$A = \sqrt{\frac{(n+1)K}{4\pi G}\rho_c^{(1-n)/n}},$$

with boundary conditions $\theta = 1$ and $\theta' = d\theta/d\xi = 0$ at $\xi = 0$. The surface $\theta = 0$ is at the radius $\xi = \xi_1$. This is the **Lane-Emden** equation.

n	γ	$\theta(\xi)$	ξ_1	$-\xi_1^2\theta_1'$	$-\xi_1/(3\theta_1')$	$[4\pi(n+1)\theta_1'^2]^{-1}$
0	∞	$1-\xi^2/6$	$\sqrt{6}$	$2\sqrt{6}$	1	$3/(8\pi)$
1	2	$\sin(\xi)/\xi$	π	π	$\pi^2/3$	$\pi/8$
3/2	5/3		3.654	2.714	5.992	0.7704
2	3/2		4.353	2.411	11.40	1.638
3	4/3		6.897	2.018	54.19	11.05
3.25	17/13		8.011	1.950	88.15	20.36
4	5/4		14.97	1.797	622.3	247.5
5	6/5	$(1+\xi^2/3)^{-1/2}$	∞	$\sqrt{3}$	∞	∞

Polytrope Solution Details

Radius:

$$R = A\xi_1$$

Mass:

$$M = 4\pi A^3 \rho_c (-\xi_1^2 \theta_1')$$

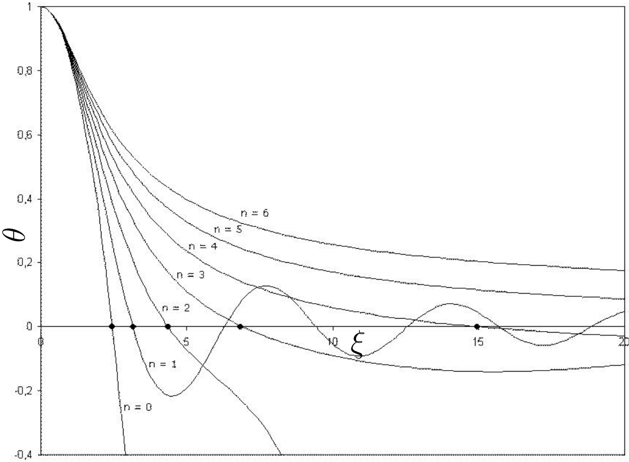
Concentration:

$$\frac{\rho_c}{\bar{\rho}} = -\frac{\xi_1}{3\theta_1'}$$

Central pressure:

$$p_c = \frac{1}{4\pi(n+1)\theta_1'^2} \frac{GM^2}{R^4}$$

Solutions become acausal when $\beta = -n\xi_1\theta_1'$.



$$K = \frac{G}{n+1}\left[4\pi\left(\frac{M}{-\xi_1^2\theta_1'}\right)^{n-1}\left(\frac{R}{\xi_1}\right)^{3-n}\right]^{1/n}, \quad R \propto K^{n/(3-n)} M^{(n-1)/(n-3)}$$

Einstein's Equations for the Schwarzschild Interior Solution

Einstein's equations for this metric are:

$$8\pi\varepsilon = \frac{1}{r^2}(1-e^{-2\Lambda}) + e^{-2\Lambda}\frac{2}{r}\frac{d\Lambda}{dr},$$
$$8\pi p = -\frac{1}{r^2}(1-e^{-2\Lambda}) + e^{-2\Lambda}\frac{2}{r}\frac{d\Phi}{dr},$$
$$\frac{dp}{dr} = -(p+\varepsilon)\frac{d\Phi}{dr}.$$

We use units such that $G=c=1$, so that $1M_\odot$ is equivalent to 1.475 km, and a pressure of 1 MeV fm^{-3} is equivalent to $1.3229\cdot 10^{-6}$ km^{-2}.

The first can be integrated to give

$$e^{-2\Lambda} = 1-2m/r, \quad \frac{dm}{dr} = 4\pi r^2\varepsilon, \quad m = 4\pi\int_0^r \varepsilon r'^2 dr' = \frac{r}{2}\ln\left(1-e^{-2\Lambda}\right).$$

The second and third combined give the so-called TOV equation:

$$-\frac{dp}{dr}\frac{1}{\varepsilon+p} = \frac{d\Phi}{dr} = \frac{m+4\pi r^3 p}{r(r-2m)}, \qquad r\le R.$$

At the center $r\to 0$, $dp/dr = d\varepsilon/dr = d\Phi/dr = d\Lambda/dr = 0$ and $m = 4\pi r^3\varepsilon_c/3 \to 0$;
at the surface $r\to R$, $p=0$, $m=M$ and $\Phi = -\Lambda = \frac{1}{2}\sqrt{1-2M/R}$.

The Uniform Density Solution in GR

Assume $\varepsilon = \varepsilon_c$. Then the inconsistency

$$p = n^2 \frac{dU}{dn} = n^2 \frac{d(\varepsilon/n - m_b)}{dn} = -\varepsilon_c$$

requires that $n = n_0$ and $U = U_0$. One finds with compactness $\beta \equiv M/R$:

$$m = \frac{4\pi}{3}\varepsilon_0 r^3, \qquad e^{\Phi} = \frac{3}{2}\sqrt{1-2\beta} - \frac{1}{2}\sqrt{1-2\beta(r/R)^2},$$

$$e^{-\Lambda} = \sqrt{1 - 2\beta\left(\frac{r}{R}\right)^2}, \qquad p = \varepsilon_c \frac{\sqrt{1-2\beta} - \sqrt{1-2\beta(r/R)^2}}{\sqrt{1-2\beta(r/R)^2} - 3\sqrt{1-2\beta}},$$

The central pressure p_c becomes infinite when $\beta \le 4/9$. This is the limiting compactness for **any** spherically symmetric star (**Buchdahl's Theorem**).

Causality requires that $p_c < \varepsilon_c$, or $\beta \le 3/8$ (although $c_s^2 = dp/d\varepsilon = \infty$ for this solution in any case.) We show later that $\beta \le 0.354$ is the real causal limit.

Assuming $U_0 = 0$, the binding energy is analytic

$$\frac{\mathrm{BE}}{M} = \frac{3}{4\beta}\left(\frac{\sin^{-1}\sqrt{2\beta}}{\sqrt{2\beta}} - \sqrt{1-2\beta}\right) - 1 \simeq \frac{3\beta}{5} + \frac{9\beta^2}{14} + \cdots$$

In the case that U_0/m_b is finite, this expansion becomes

$$\frac{\mathrm{BE}}{M} \simeq \left(1 + \frac{U_0}{m_b}\right)^{-1}\left[-\frac{U_0}{m_b} + \frac{3\beta}{5} + \frac{9\beta^2}{14} + \cdots\right].$$

$$I \simeq 0.4MR^2\left(1 - 0.87\beta - 0.3\beta^2\right)^{-1}$$

Buchdahl's Solution

A more realistic solution, with vanishing density at the surface, was discovered in 1967 by Buchdahl. It is the only known analytic solution with a specific equation of state, an extension of an $n = 1$ polytrope ($p \propto \varepsilon^2$ for $\varepsilon \to 0$):

$$\varepsilon = \sqrt{p_* p} - 5p,$$

where p_* is a parameter. Define two parametric radial variables

$$u = \frac{\beta \sin Ar'}{Ar'}, \qquad r' = \frac{r(1-2\beta)}{1-\beta+u}$$

where $A^2 = 2\pi p_* (1-2\beta)^{-1}$. Then it is found that

$$e^{2\Phi} = (1-2\beta)\frac{1-\beta-u}{1-\beta+u},$$

$$e^{2\Lambda} = \frac{(1-2\beta)(1-\beta+u)}{(1-\beta-u)(1-\beta+\beta\cos Ar')^2},$$

$$p = \frac{A^2 u^2}{8\pi}\frac{1-2\beta}{(1-\beta+u)^2} = \frac{u}{2}\frac{\varepsilon}{1-\beta-3u/2},$$

$$\varepsilon = \frac{A^2 u}{4\pi}\frac{1-2\beta}{(1-\beta-3u/2)(1-\beta+u)^2},$$

$$nm_b = \sqrt{pp_*}\left(1-4\sqrt{\frac{p}{p_*}}\right)^{3/2}, \qquad c_s^2 = \frac{dp}{d\varepsilon} = \left(\frac{1}{2}\sqrt{\frac{p_*}{p}}-5\right)^{-1}.$$

For this solution, the radius, central pressure, energy and number densities, and binding energy and moment of inertia are

$$R = (1-\beta)\sqrt{\frac{\pi}{2p_*(1-2\beta)}},$$

$$p_c = \frac{p_*}{4}\beta^2, \qquad \varepsilon_c = \frac{p_*}{2}\beta(1-5\beta/2), \qquad n_c m_b = \frac{p_*}{2}\beta(1-2\beta)^{3/2},$$

$$\frac{\mathrm{BE}}{M} = \frac{(1-1.5\beta)}{(1-\beta)\sqrt{1-2\beta}} - 1 \approx \frac{\beta}{2} + \frac{\beta^2}{2} + \frac{3\beta^3}{4} + \cdots$$

$$\frac{I}{MR^2} \simeq \left(\frac{2}{3} - \frac{4}{\pi^2}\right)\left(1 - 1.81\beta + 0.47\beta^2\right)^{-1}$$

The limit $p_c \leq \varepsilon_c$, required by causality, implies $\beta \leq 1/3$. However, the real causality condition, $c_{s,c} \leq 1$, implies $\beta \leq 1/6$.

The first of these equations is an explicit $M - R$ relation with scaling factor p_*,

$$\frac{M}{R} = \sqrt{\frac{2p_* R^2}{\pi}}\sqrt{\frac{2p_* R^2}{\pi} - 1} + 1 - \frac{2p_* R^2}{\pi}.$$

This can also be written as a quartic equation for $R(M)$.

The Newtonian limit, $\beta \to 0$, has $R = (2p_*/\pi)^{-1/2}$. As expected, this is independent of M. The causal limit has $R = (5/4)(3p_*/\pi)^{-1/2}$, which is only 2.06% larger.

Tolman VII Solution

This is a realistic solution, discovered by Tolman in 1939, in which $\varepsilon = \varepsilon_c(1-x)$ where $x = (r/R)^2$.

$$e^{-2\Lambda} = 1 - \beta x(5-3x) = 1 - \frac{2m}{r},$$

$$e^{2\Phi} = (1 - 5\beta/3)\cos^2\phi,$$

$$p = \frac{1}{4\pi R^2}\left[\sqrt{3\beta}e^{-\Lambda}\tan\phi - \frac{\beta}{2}(5-3x)\right],$$

$$n = \frac{(\varepsilon + P)}{m}\frac{\cos\phi}{\cos\phi_1},$$

$$\phi = \frac{w(x=1) - w}{2} + \phi_1,$$

$$\phi_1 = \phi(x=1) = \tan^{-1}\left(\sqrt{\frac{\beta}{3(1-2\beta)}}\right),$$

$$w = \ln\left[x - \frac{5}{6} + \frac{e^{-\Lambda}}{\sqrt{3\beta}}\right].$$

Central values of p/ε and the square of the sound speed are

$$\left(\frac{p}{\varepsilon}\right)_c = \frac{2\tan\phi_c}{5}\sqrt{\frac{1}{3\beta}} - \frac{1}{3},$$
$$c_{s,c}^2 = \tan\phi_c\left(\frac{1}{5}\tan\phi_c + \sqrt{\frac{\beta}{3}}\right).$$

There is no analytic result for the binding energy, but a series expansion is

$$\frac{\mathrm{BE}}{M} \simeq \frac{11\beta}{21} + \frac{7187\beta^2}{18018} + \frac{68371\beta^3}{306306} + \cdots.$$

In order that p_c remain finite, this solution is limited to $\phi(x=0) < \pi/2$, or $\beta < 0.3862$.

$$\frac{I}{MR^2} \simeq \frac{2}{7}\left(1 - 1.1\beta - 0.6\beta^2\right)^{-1}$$

Causality ($c_{s,c} < 1$) requires $\beta < 0.2698$.

Narai's Solution

Yet another analytic solution was discovered by Narai in 1950 using the parametric variable r',

$$r = \frac{e}{c}\frac{r'}{\cos f}\sqrt{1-2\beta}.$$

The metric functions are expressed in terms of auxiliary functions f and g:

$$e^{-\Lambda} = \sqrt{1-3\beta}\left(\frac{r'}{R'}\right)^2 \tan f,$$

$$e^{\Phi} = \sqrt{1-2\beta}\frac{e\cos g}{c\cos f},$$

$$f = \cos^{-1} e + \sqrt{\frac{3\beta}{4}}\left[1-\left(\frac{r'}{R'}\right)^2\right],$$

$$g = \cos^{-1} c + \sqrt{\frac{3\beta}{2}}\left[1-\left(\frac{r'}{R'}\right)^2\right].$$

The quantities e and c are

$$e^2 = \cos^2 f(R') = \frac{2+\beta+2\sqrt{1-2\beta}}{4+\beta/3}$$

$$c^2 = \cos^2 g(R') = \frac{2e^2}{2e^2+(1-e^2)(7e^2-3)(5e^2-3)^{-1}}.$$

Thermodynamic variables are

$$p = \frac{\cos f}{4\pi R'^2}\frac{c^2}{e^2}\sqrt{3\beta}\left\{\sqrt{2}\cos f \tan g\left[1-\sqrt{3\beta}\left(\frac{r'}{R'}\right)^2\tan f\right]-\right.$$

$$\left.-\sin f\left[2-\frac{3}{2}\sqrt{3\beta}\left(\frac{r'}{R'}\right)^2\tan f\right]\right\},$$

$$\varepsilon = \frac{\sqrt{3\beta}}{4\pi R'^2\sqrt{1-2\beta}}\frac{c^2}{e^2}\left[3\sin f\cos f-\sqrt{\frac{3\beta}{4}}\left(\frac{r'}{R'}\right)^2(3-\cos^2 f)\right],$$

$$m = \frac{r'^3}{R'^2}\frac{e}{c}\frac{\tan f}{\cos f}\sqrt{3\beta(1-2\beta)}\left[1-\sqrt{\frac{3\beta}{4}}\left(\frac{r'}{R'}\right)^2\tan f\right].$$

The pressure-density ratio and sound speed at the center are

$$\frac{p_c}{\rho_c} = \frac{1}{3}\left(\sqrt{2}\cot f(0)\tan g(0)-2\right),$$

$$c_{s,c}^2 = \frac{1}{3}\left(2\tan^2 g(0)-\tan^2 f(0)\right).$$

The central pressure and sound speed become infinite when $\cos g(0)=0$ or when $\beta=0.4126$, and the causality limit is $\beta=0.223$. This solution behaves similarly to Tolman VII.

Neutron Star Structure

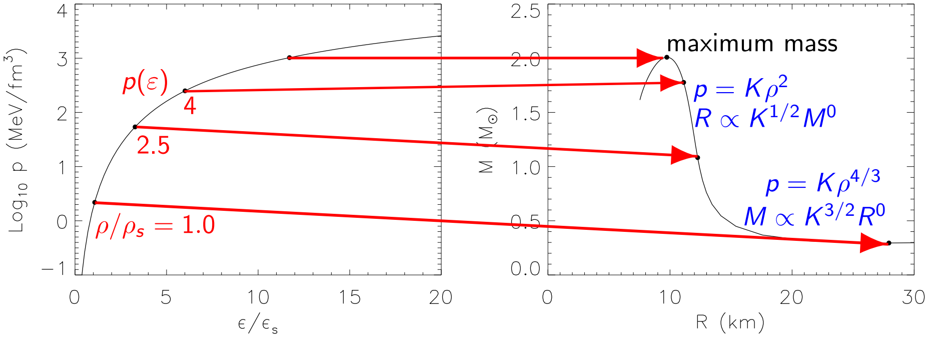
Newtonian Gravity:

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2}; \qquad \frac{dm}{dr} = 4\pi\rho r^2; \qquad \rho c^2 = \varepsilon$$

Newtonian Polytrope:

$$p = K\rho^\gamma; \qquad M \propto K^{1/(2-\gamma)} R^{(4-3\gamma)/(2-\gamma)}$$

$\rho < \rho_s$: $\gamma \simeq \frac{4}{3}$; $\qquad \rho > \rho_s$: $\gamma \simeq 2$



Mass-Radius Diagram and Theoretical Constraints

GR:

$R > 2GM/c^2$

$P < \infty$:

$R > (9/4)GM/c^2$

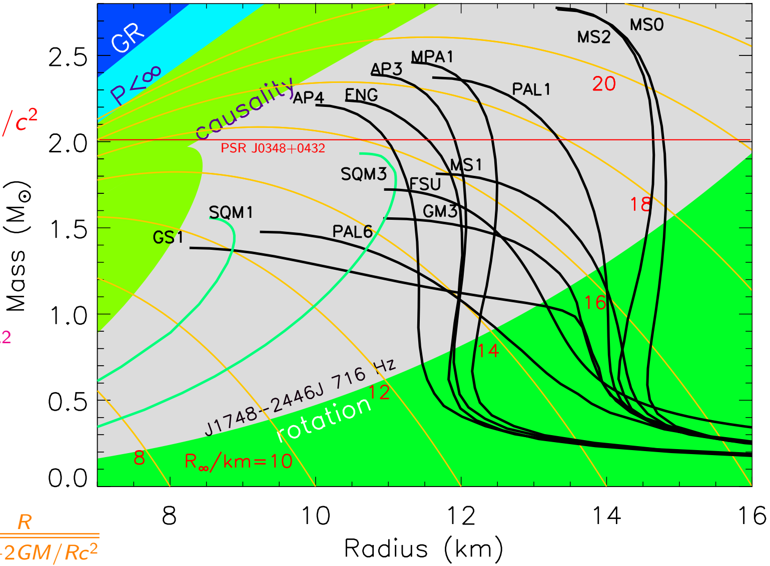
causality:

$R \gtrsim 2.9GM/c^2$

— normal NS

— SQS

— $R_\infty = \frac{R}{\sqrt{1-2GM/Rc^2}}$



Questions

- Why do radii of stars with masses $1.2M_\odot \lesssim M \lesssim 1.8M_\odot$ have such a small variation (± 0.5 km) about a **common** radius $\hat{R}$?
- Why do different equations of state predict such a wide range of common radii (9 km$\lesssim \hat{R} \lesssim$ 15 km)?

Answers:

- In the density range $\rho_s \lesssim \rho \lesssim 3\rho_s$ equations of state that satisfy the $2M_\odot$ maximum mass constraint generally predict $P \propto \rho^2$.
- The common radius seems to be determined by the pressure of neutron star matter in the density range $\rho_s \lesssim \rho \lesssim 2\rho_s$.

Pressure - Radius Correlation

Newtonian polytrope: $p = K\rho^{1+1/n}$

$$\Rightarrow R \propto K^{n/(3-n)} M^{(1-n)/(3-n)}$$

Realistic EOS: $n \simeq 1 \Rightarrow R \propto K^{1/2} M^0$

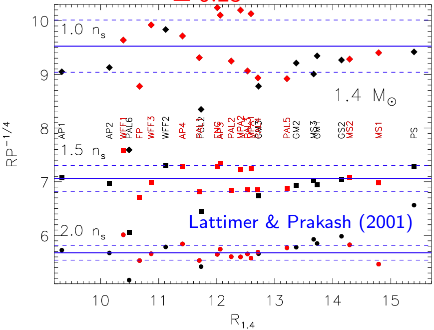
GR phenomenological result:

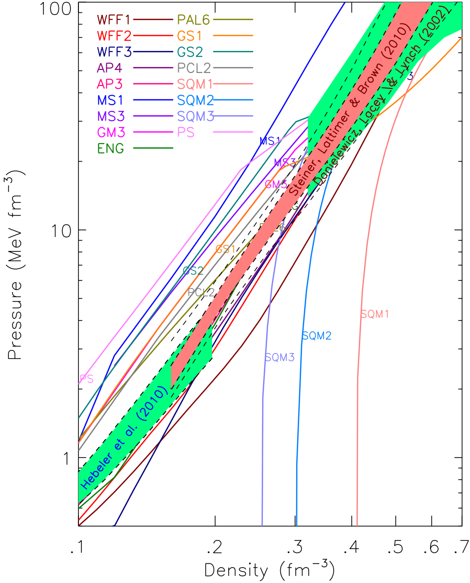
$$R \propto K^{1/4} \propto p_f^{1/4} \rho_f^{-1/2}$$

Buchdahl motivation:

$$2R^2 p_*(1-2\beta) = \pi(1-\beta)^2 \Rightarrow$$

$$\left.\frac{d\ln R}{d\ln\rho}\right|_{n,M} = \frac{(1-\beta)(1-2\beta)(\sqrt{p_*}-10\sqrt{p})}{2(1-3\beta+3\beta^2)(\sqrt{p_*}+2\sqrt{p})} \simeq 0.23$$





Lattimer & Prakash (2001)