Darmstadt Lecture 3 – Nuclear Structure

James Lattimer

Department of Physics & Astronomy
449 ESS Bldg.
Stony Brook University

July 6, 2019

Darmstadt Lecture 3 – Nuclear Structure
James.Lattimer@Stonybrook.edu

Nuclear Symmetry Energy

Defined as the difference between energies of pure neutron matter ($x=0$) and symmetric ($x=1/2$) nuclear matter.

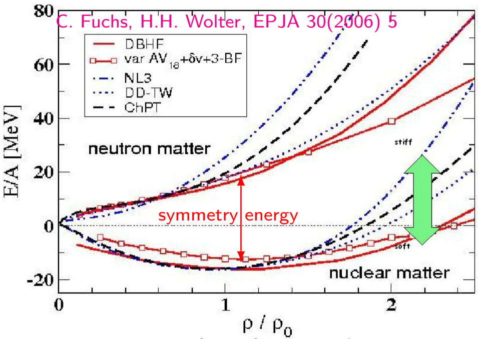
$$S(\rho) = E(\rho, x=0) - E(\rho, x=1/2)$$

Expanding around saturation density (ρ_s) and symmetric matter ($x=1/2$)

$$E(\rho, x) = E(\rho, 1/2) + (1-2x)^2 S_2(\rho) + \cdots$$

$$S_2(\rho) = \mathbf{S_v} + \frac{\mathbf{L}}{3}\frac{\rho-\rho_s}{\rho_s} + \cdots$$

$$\mathbf{S_v} \simeq 31 \text{ MeV}, \mathbf{L} \simeq 50 \text{ MeV}$$



Connections to neutron matter:

$$E(\rho_s, 0) \approx S_v + E(\rho_s, 1/2) = S_v - B, \qquad p(\rho_s, 0) = L\rho_s/3$$

Neutron star matter (in beta equilibrium):

$$\frac{\partial(E+E_e)}{\partial x} = 0, \quad p(\rho_s, x_\beta) \simeq \frac{L\rho_s}{3}\left[1 - \left(\frac{4S_v}{\hbar c}\right)^3 \frac{4-3S_v/L}{3\pi^2\rho_s}\right]$$

Nuclear Mass Formula

Bethe-Weizsäcker (neglecting pairing and shell effects)

$$E(A,Z) = -a_v A + a_s A^{2/3} + a_C Z^2/A^{1/3} + S_v A I^2.$$

Myers & Swiatecki introduced the surface asymmetry term:

$$E(A,Z) = -a_v A + a_s A^{2/3} + a_C Z^2/A^{1/3} + S_v A I^2 - S_s (N-Z)^2/A^{4/3}.$$

$a_v = B$, $a_s \simeq 18$ MeV, $a_C \simeq 0.75$ MeV, $S_s \simeq 45$ MeV, $I = (N-Z)/A$

Optimum Nucleus

$$\begin{aligned}
\frac{\partial E/A}{\partial A} &= -\frac{a_s}{3}A^{-4/3} + \frac{a_C}{6}A^{-1/3}(1-I)^2 - \frac{S_s}{3}I^2 A^{-4/3} = 0 \\
\frac{\partial E/A}{\partial I} &= \frac{-a_C}{2}A^{2/3}(1-I) + 2S_v I - 2S_s A^{-1/3} I
\end{aligned}$$

$$\begin{aligned}
A &= \frac{2(a_s + S_s I^2)}{a_C(1-I)^2} \sim 48(1-I)^{-2} \sim 57 \\
I &= \frac{a_C}{4S_v A^{-2/3} - 2S_s/A + a_C} \sim \frac{1}{12}
\end{aligned}$$

Nuclei at High Densities From Liquid Drop

$$E(Z,N) \simeq -BA + S_v A I^2 + (E_s - S_s I^2)A^{2/3} + E_C \frac{Z^2}{A^{1/3}}$$

$B \simeq 16$ MeV, $S_v \simeq 30$ MeV, $E_s \simeq 18$ MeV, $S_s \simeq 45$ MeV, $E_C \simeq 0.75$ MeV.

At each density, the preferred nucleus has a mass determined by

$$\left(\frac{\partial(E/A)}{\partial A}\right)_x = -\frac{E_s - S_s I^2}{3A^{4/3}} + \frac{2E_C x^2}{3A^{1/3}} = 0$$

. The Nuclear Virial Theorem is

$$E_s - S_s I^2 = 2E_C x^2 A, \qquad A_{opt} = 2\frac{E_s - S_s I^2}{E_C(1-I)^2} \simeq 48(1+2I) \simeq 61.$$

At low densities, the optimum nucleus has a charge determined by

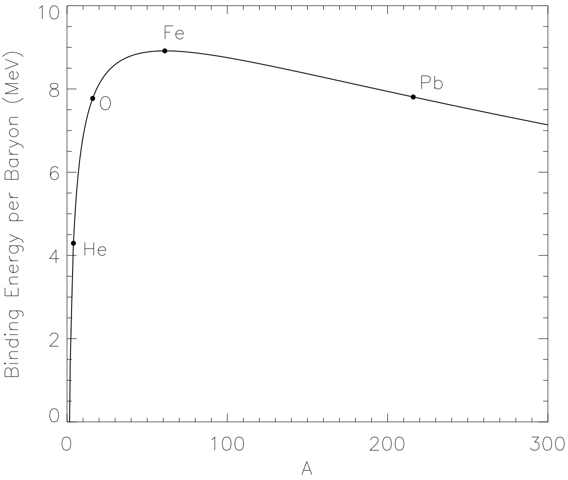
$$\left(\frac{\partial(E/A)}{\partial x}\right)_A = -4I\left(S_v - \frac{S_s}{A^{1/3}}\right) + (1-I)E_C A^{2/3} = 0,$$

$$I = \frac{E_C A}{4(S_v A^{1/3} - S_s) + E_C A} \simeq 0.125; \qquad Z \simeq 27$$

Isolated Nuclei

The binding energy curve is heavily skewed. Certain closed-shell nuclei (He, C, O, Pb) have much larger binding than the average.

The optimum value of I increases with mass number A. This trend represents the *Valley of Beta Stability*.



Nuclei at Higher Densities

At the end of stellar evolution, when an iron core forms, the central stellar density is about $\rho \simeq 10^7$ g cm^{-3}, implying a filling factor $u = \rho/\rho_s \simeq 3.7 \cdot 10^{-8}$. The intranuclear spacing is about $2u^{-1/3} \simeq 600$ nuclear radii.

Electron screening reduces the nuclear Coulomb energy.

Approximating electrons as uniformly distributed, even within nuclei, the nuclear Coulomb energy is:

$$E_{Coul} = \frac{3}{5}\frac{Z^2 e^2}{R}\left(1 - \frac{3}{2}u^{1/3} + \frac{u}{2}\right)$$

The reduction factor is about 0.5% for $\rho \simeq 10^7$ g cm^{-3}.

This effect increases the nuclear mass, which is proportional to E_{Coul}^{-1}, as the average density increases.

The optimum I also increases with density due to *beta equilibrium*:

$$\frac{\partial(E/A + E_e)}{\partial x} = -\mu_n + \mu_p + \mu_e = 0.$$

Chemical Potentials and Neutron Drip

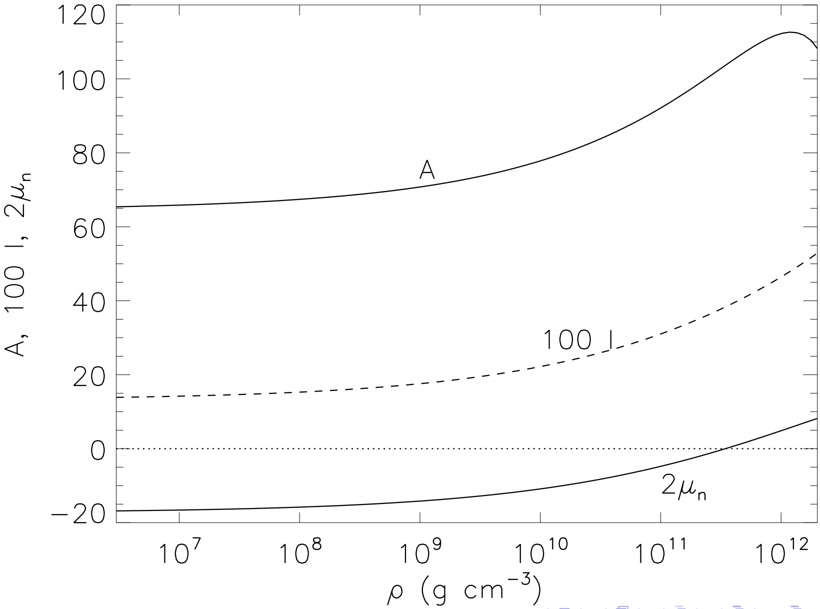
Chemical potentials are equivalent to the separation energies:

$$\mu_n = \left(\frac{\partial(E/A)}{\partial N}\right)_Z, \qquad \mu_p = \left(\frac{\partial(E/A)}{\partial Z}\right)_N,$$

$$\mu_e = \frac{\partial(E_e n Y_e)}{\partial(nY_e)} = \hbar c(3\pi^2 n_s u x)^{1/3}, \qquad Y_e = x,$$

$$\mu_n - \mu_p = -\left(\frac{\partial(E/A)}{\partial x}\right)_A,$$

At sufficiently high density, about $\rho = (3.5 - 4) \cdot 10^{11}$ g cm^{-3}, as x becomes smaller and A becomes larger, μ_n becomes positive. Neutrons thus 'drip' out of nuclei.

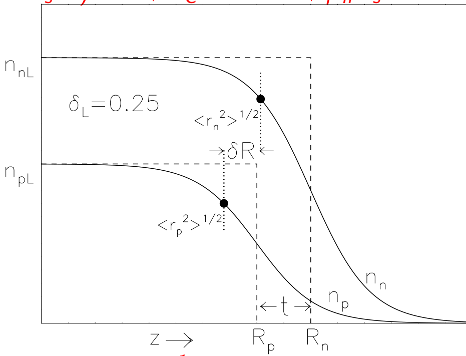

A, 100 I, $2\mu_n$
120
100
80
60
40
20
0
−20
A
100 I
$2\mu_n$
10^7
10^8
10^9
10^{10}
10^{11}
10^{12}
ρ (g cm^{-3})

Nuclear Droplet Model

Myers & Swiatecki droplet extension: consider the variation of the neutron/proton asymmetry within the nuclear surface.

$$E(A,Z) = (-B + S_v\delta^2)(A - N_s) + (E_s - S_s\delta^2)A^{2/3} + E_C Z^2 A^{-1/3} + \mu_n N_s.$$

N_s is the number of excess neutrons associated with the surface, $\delta = 1 - 2x = (A - N_s - 2Z)/(A - N_s)$ is the asymmetry of the nuclear bulk fluid, and $\mu_n = -a_v + S_v\delta(2-\delta)$ is the neutron chemical potential. Surface tension is the surface thermodynamic potential; adding $\mu_n N_s$ gives the total surface energy. Optimizing $E(A,Z)$ with respect to N_s yields



$$N_s = \frac{S_s}{S_v}\frac{\delta}{1-\delta} = A\frac{I-\delta}{1-\delta}, \quad \delta = I\left(1 + \frac{S_s}{S_v A^{1/3}}\right)^{-1},$$

$$E(A,Z) = -BA + E_s A^{2/3} + E_C Z^2/A^{1/3} + S_v A I^2 \left(1 + \frac{S_s}{S_v A^{1/3}}\right)^{-1}.$$

Macroscopic Hydrodynamic Nuclear Model

$$\mathcal{H} = \mathcal{H}_B(n,\alpha)+\mathcal{H}_C(n,\alpha)+\mathcal{Q}(n)(n')^2, \quad \mathcal{H}_B(n,\alpha) = \mathcal{H}_B(n,0)+v_{sym}(n)\alpha^2$$

$$\alpha = n_n - n_p, \qquad n' = dn/dr, \qquad v_{sym} = S_2(n)/n, \qquad \mathcal{H}_\mathcal{C} = n_p V_C/2$$

$$V_C(r) = \frac{e^2}{r}\int_0^r n_p(r')d^3r' + \int_r^\infty \frac{e^2}{r'} n_p(r')d^3r' \simeq \frac{Ze^2}{R}\left(\frac{3}{2}-\frac{r^2}{2R^2}\right)$$

Optimize total energy with respect to n and α subject to fixed $A = \int \rho d^3r$ and $N - Z = \int \alpha d^3r$:

$$\frac{\delta}{\delta n}[\mathcal{H} - \mu n] = 0, \qquad \frac{\delta}{\delta \alpha}[\mathcal{H} - \bar{\mu}\alpha] = 0.$$

$$\mu = \frac{\partial[\mathcal{H}_\mathcal{B}+\mathcal{H}_C]}{\partial n} = 2\frac{d}{dr}[\mathcal{Q}n'] + \frac{\partial \mathcal{Q}}{\partial n}(n')^2, \qquad \bar{\mu} = \frac{\partial[\mathcal{H}_B+\mathcal{H}_C]}{\partial \alpha}$$

Multiply first by n' and second by α' and add, then integrate once:

$$\mathcal{Q}(n)(n')^2 = \mathcal{H}_B + \mathcal{H}_C - \mu n - \bar{\mu}\alpha$$

Surface Energy

Consider a symmetric, chargeless nucleus. The total energy, the sum of volume and surface energies, is

$$\int \mathcal{H} d^3r = \mu A + \int \left[\mathcal{H} - \mu n\right] d^3r \simeq \mu A + 4\pi R^2 \int_{-\infty}^{\infty} \left[\mathcal{H} - \mu n\right] dx$$

We made a *leptodermous expansio*n above to isolate the surface.

$$E_{vol,o} + E_{surf,o} = \mu A + 4\pi R^2 \int \left[\mathcal{H}_B(n,0) - \mu n + \mathcal{Q}(n)(n')^2\right] dx$$

Assume $\mathcal{H}_B(n,0) = n\mu + (nK_s/18)(1-u)^2$, $u = n/n_s$, $\mathcal{Q}(n) = Q/n$, $\mu = -B$:

$$\frac{du}{dz} = -u(1-u), \quad z = \frac{x}{a}, \quad a = \sqrt{\frac{18Q}{K_s}}, \qquad u = \frac{1}{1+e^{z-y}}$$

$$A = 4\pi n_s a^3 F_2(y), \quad F_i(y) = \int_0^\infty \frac{z^i \, dz}{1+e^{z-y}} \simeq \frac{y^{i+1}}{i+1}\left[1 + \frac{i(i+1)}{6}\left(\frac{\pi}{y}\right)^2 + \dots\right]$$

This determines the parameter $y \approx R/a$, which is related to the radius.
The surface tension for symmetric matter is

$$\sigma_o = \frac{E_{surf,o}}{4\pi R^2} = 2Q \int_0^\infty \left(\frac{dn}{dz}\right)^2 \frac{dz}{n} = 2Q \int_{n_s}^0 \frac{dn}{dz}\frac{dn}{n} = \frac{Qn_s}{a}$$

Parameter determination:

$$t_{90-10} = a\int_{0.1}^{0.9} \frac{dz}{du} du = 4a \ln 3 = 2.3 \text{ fm}, \quad a \simeq 0.523 \text{ fm}$$

$$Q = \frac{K_s}{18}\left(\frac{t_{90-10}}{4\ln 3}\right)^2 \simeq 3.65 \text{ MeV fm}^2, \qquad K_s = 240 \text{ MeV}$$

$$\sigma_o = \int [\mathcal{H} - \mu n] dx = 2Q \int_0^\infty \frac{(n')^2}{n} dx = 2Q \int_{n_s}^0 \frac{n'}{n} dn$$
$$= \frac{Q n_s}{a} \simeq 1.17 \text{ MeV fm}^{-2}$$

$$E_s = 4\pi r_o^2 \sigma_o \simeq 19.2 \text{ MeV}, \qquad r_o = \left(\frac{3}{4\pi n_s}\right)^{1/3} \simeq 1.143 \text{ fm}$$

Determination of Isovector Density α

$$\alpha = \frac{\bar{\mu} + \partial\mathcal{H}_C/\partial\alpha}{2v_{sym}} = \frac{\bar{\mu} + V_C/2}{v_{sym}}$$

$$N - Z = IA = \int \frac{\bar{\mu} + V_C/2}{2v_{sym}} d^3r \equiv \frac{\bar{\mu}}{2} H_0 + \frac{G}{4}$$

$$H_i = \int \left(\frac{r}{R}\right)^i \frac{d^3r}{v_{sym}}, \qquad G = \frac{V_C}{v_{sym}} d^3r = \frac{Ze^2}{2R}(3H_0 - H_2)$$

$$\alpha = \frac{N - Z - (G - V_C H_0)/4}{v_{sym} H_0}$$

$$\alpha_o = \frac{n_s}{S_v H_0}\left(N - Z + \frac{Ze^2}{8R} H_2\right), \qquad v_{sym}(n_s) = \frac{S_v}{n_s}$$

Polarization results in an increase in asymmetry near the center.

Nuclear Structure

Total symmetry and Coulomb energy:

$$E_{sym}+E_C=\int v_{sym}\alpha^2 d^3r+\frac{1}{4}\int(n-\alpha)V_C d^3r$$
$$=\frac{(N-Z)^2}{H_0}-\frac{(N-Z)G}{4H_0}+\frac{1}{4}\int nV_C d^3r$$
$$=\frac{(N-Z)^2}{H_0}+\frac{3}{5}\frac{Z^2e^2}{R}+\frac{Ze^2}{8R}(N-Z)\left(\frac{H_s}{H_0}-\frac{3}{5}\right)$$

Dipole polarizability α_D:

$$\frac{\delta}{\delta\alpha}\left(\int\mathcal{H}d^3r-\epsilon\int z\alpha d^3r\right)=0$$

α_d is the function $\alpha(r)$ which solves this.

$$\alpha_d=\frac{\epsilon z+V_C}{2v_{sym}}$$

$$\alpha_D=\frac{1}{2\epsilon}\int z\alpha_d d^3r=\frac{1}{4\epsilon}\int z\frac{\epsilon z+V_C}{v_{sym}}d^3r=\frac{R^2H_2}{12}$$

Neutron Skin Thickness

$$\frac{4\pi}{3}\left(R_n^3 - R_p^3\right) = \int\left(\frac{n_n}{n_{no}} - \frac{n_p}{n_{po}}\right) d^3r = \frac{n_s}{2n_{no}n_{po}}\int\left(\alpha - n\frac{\alpha_o}{n_s}\right) d^3r$$
$$= \frac{n_s A}{2n_{no}n_{po}}\left(I - \frac{\alpha_o}{n_s}\right)$$

$$\frac{R_n - R_p}{R} \simeq \frac{2}{3}\frac{I - \alpha_o/n_s}{1 - \alpha_o^2/n_s^2} = \frac{2}{3}\left[I\left(1 - \frac{A}{S_v H_0}\right) - \frac{Ze^2}{8RS_v}\frac{H_2}{H_0}\right]$$

Polarization decreases neutron skin thickness.
Symmetric nuclei have a proton skin.

$$r_{n,p}^2 = \frac{1}{(N,Z)}\int n_{n,p} r^2 d^3r = \frac{1}{2(N,Z)}\int (n \pm \alpha) r^2 d^3r$$
$$= \frac{R^2}{1 \pm I}\left[\frac{3}{5} \pm \left(I\frac{H_2}{H_0} - \frac{Ze^2}{8RA}\left[H_4 - \frac{H_2^2}{H_0}\right]\right)\right]$$

$$\frac{r_{np}}{R} \equiv \frac{r_n - r_p}{R} = \sqrt{\frac{3}{5}}\left[I\left(\frac{5H_2}{3H_0} - 1\right) - \frac{5Ze^2}{24RA}\left(H_4 - \frac{H_2^2}{H_0}\right)\right]\frac{1}{\sqrt{1 - I^2}}$$

Solutions for Arbitrary v_{sym}

Assume v_{sym} can be expanded as

$$\frac{S_v}{n_s v_{sym}(u)} = \sum_{j=1}^{J} b_j u^j, \qquad 0 < u < 1$$

If the density has the Fermi shape, we find

$$H_i = \int \left(\frac{r}{R}\right)^i \frac{d^3 r}{v_{sym}} = \frac{4\pi n_s a^{3+i}}{S_v R^i}\left[F_{i+2}(y) - (2+i)\mathcal{T} F_{i+1}(y) + \cdots\right]$$

$$\mathcal{T} = b_2 + \frac{3}{2} b_3 + \frac{11}{5} b_4 + \frac{25}{12} b_5 + \frac{137}{60} b_6 + \cdots$$

$\sum_{j=1}^{J} b_j = 1$ ensures that $v_{sym}(u=1) = S_v/n_s$.
Noting that in the droplet model $E_{sym} = S_v A I^2 (1 + S_s A^{-1/3}/S_v)^{-1}$ and in this model $E_{sym} = A^2 I^2 / H_0$, identify

$$H_0 = \frac{A}{S_v}\left(1 + \frac{S_s}{S_v A^{1/3}}\right) \simeq \frac{A}{S_v}\left(1 - \frac{3\mathcal{T} a}{r_o A^{1/3}}\right)$$

so that $S_s \simeq -3\mathcal{T} S_v a / r_o$.

It now follows that

$$H_i \simeq \frac{A}{S_v}\left(\frac{3}{3+i}+\frac{S_s}{S_v A^{1/3}}\right)$$

$$\alpha_D \simeq \frac{AR^2}{20S_v}\left(1+\frac{5}{3}\frac{S_s}{S_v A^{1/3}}\right)$$

$$r_{np} \simeq \sqrt{\frac{3}{5(1-I^2)}}\frac{2r_o}{3}\left(1+\frac{S_s}{S_v A^{1/3}}\right)^{-1}\left[I\frac{S_s}{S_v}-\frac{3Ze^2}{140r_o S_v}\left(1+\frac{10}{3}\frac{S_s}{S_v A^{1/3}}\right)\right]$$

As an example, consider the expansion

$$S_2(u) \simeq S_v + \frac{L}{3}(u-1)+\frac{K_{sym}}{18}(u-1)^2+\cdots, \qquad K_{sym} \simeq 18\left(\frac{L}{3}-S_v\right)$$

To leading order, one finds $S_s/S_v \simeq 1.35$

$$b_1 = 1+\frac{L}{3S_v}+\left(\frac{L}{3S_v}\right)^2-\frac{K_{sym}}{18S_v} \simeq 2.37$$

$$b_2 = \frac{K_{sym}}{9S_v}-\frac{L}{3S_v}-2\left(\frac{L}{3S_v}\right)^2 \simeq -2.14 \quad b_3 = \left(\frac{L}{3S_v}\right)^2-\frac{K_{sym}}{18S_v} \simeq -0.76$$

$$\mathcal{T} = -\frac{L}{3S_v}-\frac{1}{2}\left(\frac{L}{3S_v}\right)^2+\frac{K_{sym}}{36S_v} = -\frac{1}{2}\left[1+\frac{L}{3S_v}+\left(\frac{L}{3S_v}\right)^2\right] \simeq -0.992$$