Evgeny Epelbaum, RUB

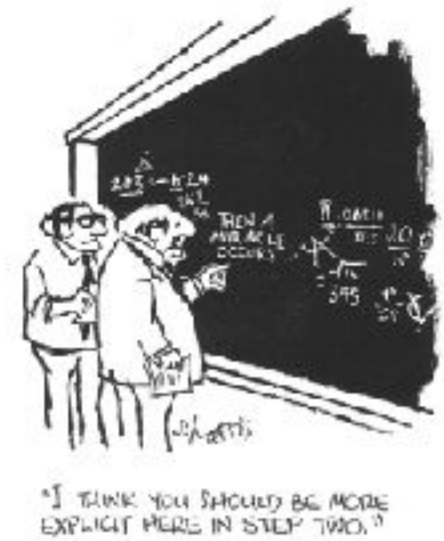
Lecture Week, SFB 1245
TU Darmstadt, September 26-30, 2022

Chiral effective field theory for nuclear forces

Summary day 3

— Effective range expansion, pionless EFT, naturalness and fine tuning, power counting and renormalization conditions, RG analysis, explicit and implicit renormalization, …

Today: Inclusion of pions



Outline day 4

Part I: Concepts and the framework

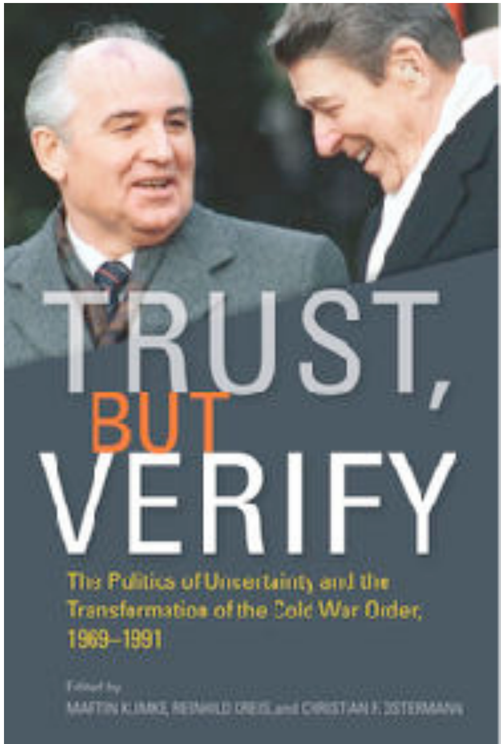
— Are pions perturbative? How to test the long-range dynamics?

1. Low-energy theorems (LETs) and the modified ERE
2. KSW with perturbative pions
3. Non-perturbative inclusion of pions
4. How not to renormalize the Schrödinger eq.

Part II: Methods

— how to derive nuclear forces & currents?

1. Introduction
2. Method of Unitary Transformation
3. Merging MUT with ChPT
4. Example: NLO correction to the NN force
5. A note on renormalization



Modified Effective Range Expansion (MERE)

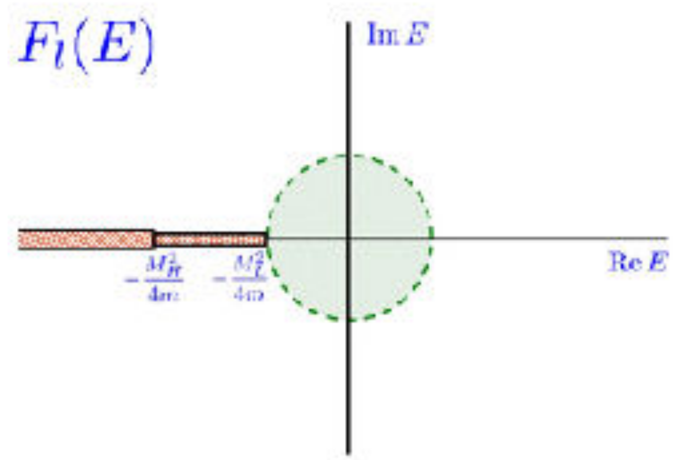
1. LETs and the MERE

What are the low-energy theorems?

Two-range potential: $V(r) = V_L(r) + V_S(r)$

with $M_L^{-1} \gg M_H^{-1}$

- $F_l(k^2)$ is meromorphic in $|k| < M_L/2$



- $$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot\left[\delta_l(k) - \delta_l^L(k)\right]$$

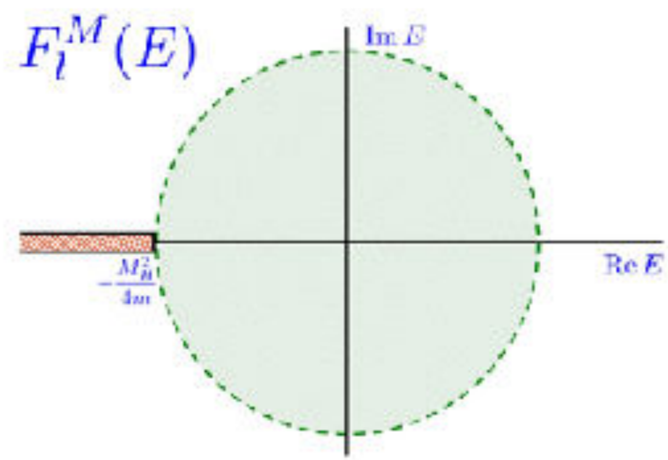
← modified effective range function
van Haeringen, Kok '82

$$f_l^L(k) = \lim_{r \to 0} \left(\frac{l!}{(2l)!} (-2ikr)^l f_l^L(k, r) \right)$$

Jost function for $V_L(r)$ — Jost solution for $V_L(r)$

$$M_l^L(k) = Re\left[\frac{(-ik/2)^l}{l!} \lim_{r \to 0} \left(\frac{d^{2l+1}}{dr^{2l+1}} \frac{r^l f_l^L(k,r)}{f_l^L(k)} \right) \right]$$

Per construction, F_l^M reduces to F_l for $V_L = 0$ and is meromorphic in $|k| < M_H/2$



MERE and low-energy theorems

Example: proton-proton scattering

$$F_C(k^2) = C_0^2(\eta)\, k \cot[\delta(k) - \delta^C(k)] + 2k\,\eta\, h(\eta) = -\frac{1}{a^M} + \frac{1}{2} r^M k^2 + v_2^M k^4 + \ldots$$

where $\delta^C \equiv \arg\Gamma(1 + i\eta)$, $\eta = \frac{m}{2k}\alpha$, $C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$, $h(\eta) = \mathrm{Re}\left[\Psi(i\eta)\right] - \ln(\eta)$

Coulomb phase shift — Sommerfeld factor — Digamma function $\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$

MERE and low-energy theorems

Long-range forces impose correlations between the ER coefficients (low-energy theorems)
[Cohen, Hansen '99; Steele, Furnstahl '00]

The emergence of the LETs can be understood in the framework of MERE:

$$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot\left[\delta_l(k) - \delta_l^L(k)\right]$$

meromorphic for $k^2 < (M_H/2)^2$

can be computed if the long-range force is known

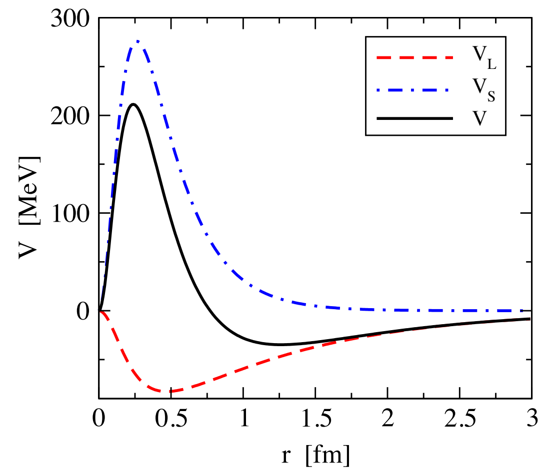
– approximate $F_l^M(k^2)$ by first 1,2,3,… terms in the Taylor expansion in k^2
– calculate all "soft" quantities
– reconstruct $\delta_l^L(k)$ and predict all coefficients in the ERE

Toy model: Low-energy theorems

$$V(r) = \underbrace{v_L e^{-M_L r} f(r)}_{V_L} + \underbrace{v_H e^{-M_H r} f(r)}_{V_H}$$

where $f(r) = \frac{(M_H r)^2}{1 + (M_H r)^2}$

and $M_L = 1.0\,,\ v_L = -0.875\,,\ M_H = 3.75\,,\ v_H = 7.5$ (all in fm^{-1})



ERE and MERE

	a	r	v_2	v_3	v_4
F_0 [fm^n]	5.458	2.432	0.113	0.515	−0.993
F_0^M [M_S^{-n}]	1.710	−1.063	−0.434	−0.680	2.624

Toy model: Low-energy theorems

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ERE and MERE

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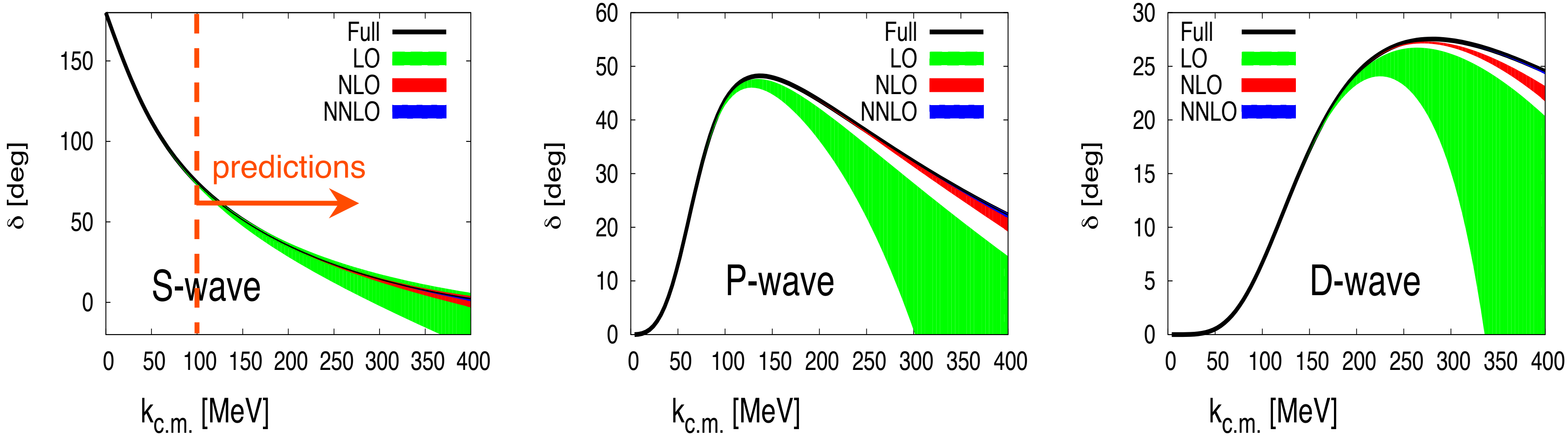
for an analytic example, see EE, Gegelia, EPJ A41 (2009) 341

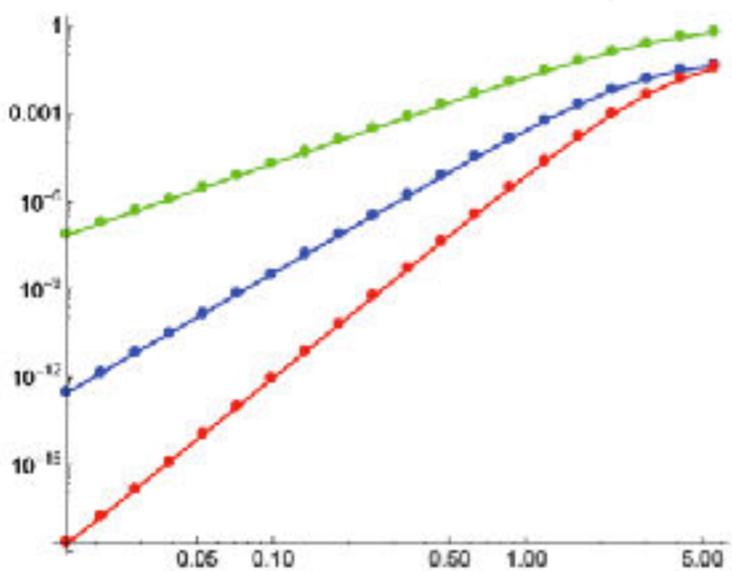
Low-Energy Theorems

	LO	NLO	NNLO	"Exp"
r	2.447(38)	2.432197161	2.432197161	2.432197161
v_2	0.12(11)	0.1132(29)	0.112815751	0.112815751
v_3	0.61(12)	0.517(16)	0.51533(20)	0.51529
v_4	$-0.95(5)$	$-0.991(14)$	$-0.9925(11)$	-0.9928

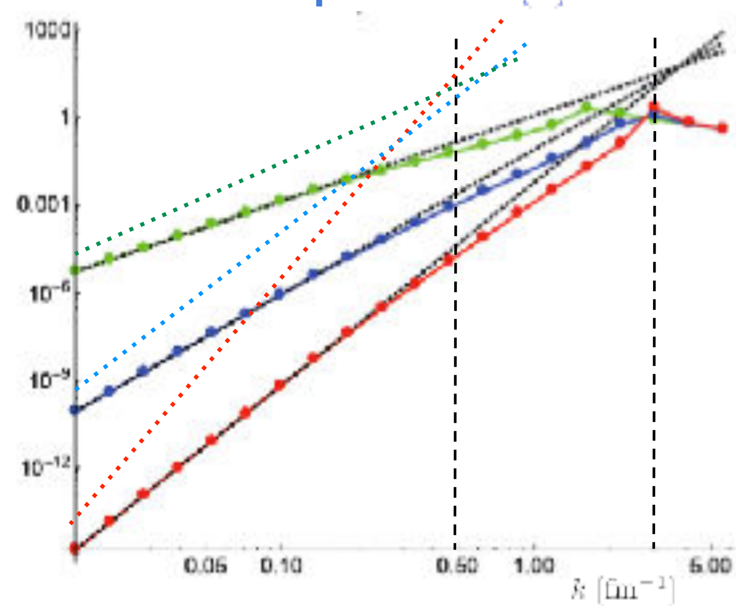
Toy model: phase shifts & error plots



Error plots for $\delta^M(k)$



Error plots for $\delta(k)$



Chiral EFT for NN scattering

2. KSW with perturbative pions

Recall the differences between the W and KSW counting schemes:

- **Weinberg:** $\mu \sim \mathcal{O}(1), \;\; \mu_i \sim \mathcal{O}(p)$ $\longrightarrow$ $V^{\rm LO}_{\rm Weinberg} \sim \mathcal{O}(1), \;\; V^{\rm NLO}_{\rm Weinberg} \sim \mathcal{O}(p^2)$

  [i.e. scaling of C_{2n} according to NDA (~ O(1))]

- **KSW:** $\mu, \mu_i \sim \mathcal{O}(p)$ $\longrightarrow$ $V^{\rm LO}_{\rm KSW} \sim \mathcal{O}(p^{-1}), \;\; V^{\rm NLO}_{\rm KSW} \sim \mathcal{O}(1)$

  [i.e. scaling of C_{2n} as $C_{2n} \sim O(p^{-1-n})$]

While the two schemes are equivalent for pionless theory, they suggest different scenarios for pionful (chiral) EFT:

$$V_{1\pi} = -\left(\frac{g_A}{2F_\pi}\right)^2 \frac{\vec\sigma_1 \cdot \vec q \, \vec\sigma_2 \cdot \vec q}{q^2 + M_\pi^2} \, \vec\tau_1 \cdot \vec\tau_2 \sim \mathcal{O}(1)$$

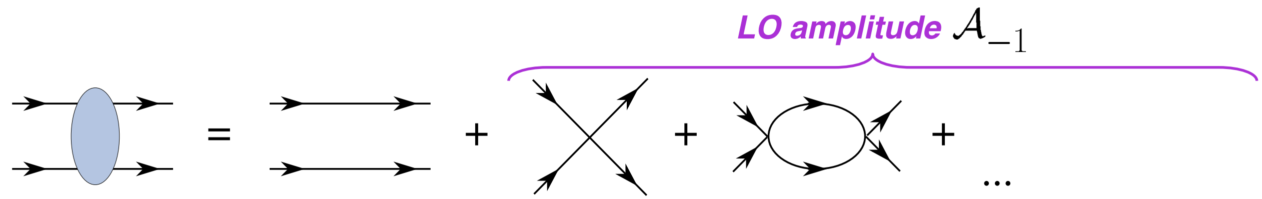
OPE is expected to be:

— LO contribution (nonperturbative) in the Weinberg scheme,

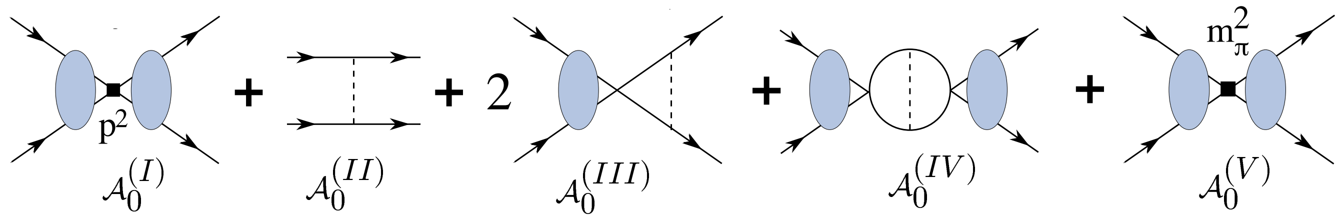
— NLO contribution (perturbative) in the KSW scheme.

Chiral EFT for NN: The KSW approach

- **Leading order:**



- **NLO:** $\mathcal{A}_0$ =



$$\mathcal{A}_0^{(I)} = -C_2^{(^1S_0)} p^2 \left[\frac{\mathcal{A}_{-1}}{C_0^{(^1S_0)}} \right]^2, \qquad \mathcal{A}_0^{(II)} = \left(\frac{g_A^2}{2f^2} \right) \left(-1 + \frac{m_\pi^2}{4p^2} \ln \left(1 + \frac{4p^2}{m_\pi^2} \right) \right)$$

(f → $\sqrt{2} F_\pi$)

$$\mathcal{A}_0^{(III)} = \frac{g_A^2}{f^2} \left(\frac{m_\pi M \mathcal{A}_{-1}}{4\pi} \right) \left(-\frac{(\mu + ip)}{m_\pi} + \frac{m_\pi}{2p} \left[\tan^{-1} \left(\frac{2p}{m_\pi} \right) + \frac{i}{2} \ln \left(1 + \frac{4p^2}{m_\pi^2} \right) \right] \right)$$

$$\mathcal{A}_0^{(IV)} = \frac{g_A^2}{2f^2} \left(\frac{m_\pi M \mathcal{A}_{-1}}{4\pi} \right)^2 \left(-\left(\frac{\mu + ip}{m_\pi} \right)^2 + \left[i \tan^{-1} \left(\frac{2p}{m_\pi} \right) - \frac{1}{2} \ln \left(\frac{m_\pi^2 + 4p^2}{\mu^2} \right) + 1 \right] \right)$$

$$\mathcal{A}_0^{(V)} = -D_2^{(^1S_0)} m_\pi^2 \left[\frac{\mathcal{A}_{-1}}{C_0^{(^1S_0)}} \right]^2$$

For more details see:
Kaplan, Savage, Wise, Nucl. Phys. B534 (1998) 329.

LETs for NN S-waves

Use these results to test the LETs for S-waves: [Cohen, Hansen, PRC 59 (1999) 13]

$$p\cot\delta_0(p) = \frac{4\pi}{m}\left[\frac{1}{\mathcal{A}_{-1}} - \frac{\mathcal{A}_0}{(\mathcal{A}_{-1})^2} + \ldots\right] + ip \stackrel{!}{=} -\frac{1}{a} + \frac{1}{2}rp^2 + v_2p^4 + v_3p^6 + v_4p^8 + \ldots$$

Express the LECs C_0, C_2, in terms of a and r to predict the shape parameters, e.g.:

$$v_2 = \frac{g_A^2 m}{16\pi F_\pi^2}\left(-\frac{16}{3a^2M_\pi^4} + \frac{32}{5aM_\pi^3} - \frac{2}{M_\pi^2}\right), \qquad v_3 = \frac{g_A^2 m}{16\pi F_\pi^2}\left(\frac{16}{a^2M_\pi^6} - \frac{128}{7aM_\pi^5} + \frac{16}{3M_\pi^4}\right), \ldots$$

$^1S_0$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
NLO KSW Cohen, Hansen '99	fit	fit	-3.3	18	-108
Nijmegen PWA	-23.7	2.67	-0.5	4.0	-20

$^3S_1$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
NLO KSW Cohen, Hansen '99	fit	fit	-0.95	4.6	-25
Nijmegen PWA	5.42	1.75	0.04	0.67	-4.0

→ large deviations suggest that pions should be treated nonperturbatively…

[even stronger evidence comes from phase shifts at N²LO, see: Fleming, Mehen, Stewart, NPA 677 (2000) 313]

Nonperturbative inclusion of pions

3. Nonperturbative inclusion of pions

LO scattering amplitude:

$$T(\vec{p}\,', \vec{p}) \;=\; \left[V_{\rm cont}(\vec{p}\,', \vec{p}) + V_{1\pi}(\vec{p}\,', \vec{p})\right] + m \int \frac{d^3 l}{(2\pi)^3} \frac{\left[V_{\rm cont}(\vec{p}\,', \vec{l}\,) + V_{1\pi}(\vec{p}\,', \vec{l}\,)\right] T(\vec{l}, \vec{p})}{p^2 - l^2 + i\epsilon}$$

Complications (as compared to pionless theory):

— $V_{1\pi}$ is not separable, no analytic results beyond 2 loops are available,

— $1/r^3$ singularity of $V_{1\pi}$

Static OPEP in coordinate space:

tensor operator: $S_{12} = 3\,\vec{\sigma}_1 \cdot \hat{r}\,\vec{\sigma}_2 \cdot \hat{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2$

$$V_{1\pi}(\vec{r}\,) = \left(\frac{g_A}{2F_\pi}\right)^2 \tau_1 \cdot \tau_2 \left[M_\pi^2 \frac{e^{-M_\pi r}}{12\pi r}\left(S_{12}(\hat{r})\left(1 + \frac{3}{M_\pi r} + \underbrace{\frac{3}{(M_\pi r)^2}}\right) + \vec{\sigma}_1 \cdot \vec{\sigma}_2\right) - \frac{1}{3}\,\vec{\sigma}_1 \cdot \vec{\sigma}_2\,\delta^3(r)\right]$$

singular potential in all S=1 channels
(solutions to the Schröd./LS equation still exist in repulsive cases)

→ Need counter terms in all spin-triplet waves! In fact, infinitely many c.t.'s are needed in every spin-triplet channel to remove UV divergences from iterations…

Nonperturbative inclusion of pions

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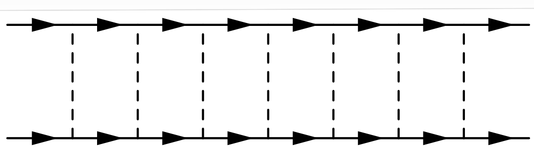
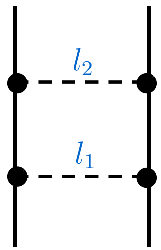
$$V_{1\pi}(\vec{r}\,) = \left(\frac{g_A}{2F_\pi}\right)^2 \tau_1\cdot\tau_2 \left[M_\pi^2 \frac{e^{-M_\pi r}}{12\pi r}\left(S_{12}(\hat{r})\left(1 + \frac{3}{M_\pi r} + \underbrace{\frac{3}{(M_\pi r)^2}}\right) + \vec{\sigma}_1\cdot\vec{\sigma}_2\right) - \frac{1}{3}\,\vec{\sigma}_1\cdot\vec{\sigma}_2\,\delta^3(r)\right]$$

singular potential in all S=1 channels
(solutions to the Schröd./LS equation still exist in repulsive cases)

E.g.:  $\propto \dfrac{1}{d-4}\,\vec{p}^{\,6}\, m_N^6$ (spin-triplet)

Re-summation of ladder diagrams

Certain terms in the amplitude must be re-summed (ladder-type graphs enhanced Weinberg '90, '91)



$$\int \frac{d^4 l}{(2\pi)^4} \frac{i(2m_N)^2\, l_1^i l_1^j l_2^k l_2^l}{[(p-l)^2 - m_N^2 + i\epsilon][(p+l)^2 - m_N^2 + i\epsilon][l_1^2 - M_\pi^2 + i\epsilon][l_2^2 - M_\pi^2 + i\epsilon]} \xrightarrow{\text{NDA}} \mathcal{O}(Q^2)$$

$$= \int \frac{d^3 l}{(2\pi)^3}\left[\underbrace{\frac{l_1^i l_1^j}{\omega_1^2}\left(\frac{m_N}{\vec{p}^{\,2} - (\vec{p} - \vec{l})^2 + i\epsilon}\right)\frac{l_2^k l_2^l}{\omega_2^2}} + \underbrace{\frac{\omega_1^2+\omega_1\omega_2+\omega_2^2}{2\omega_1^3\omega_2^3(\omega_1+\omega_2)}\, l_1^i l_1^j l_2^k l_2^l} + \mathcal{O}\left(\frac{1}{m_N}\right)\right]$$

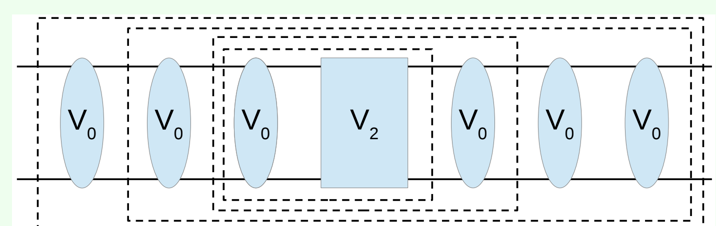
$V_{1\pi}\, G_0\, V_{1\pi}$: enhanced by the factor of m_N/Q

irreducible (static) two-pion exchange: scales according to NDA, i.e. $\sim \mathcal{O}(Q^2)$

Divergent integrals in the Lippmann-Schwinger equation are usually regularized with a cutoff Λ:

- the „RG invariant" approach with $\Lambda \gg \Lambda_b$: $T \sim 1 + \Lambda + \Lambda^2 + \ldots = (1-\Lambda)^{-1}$ van Kolck, Long, Yang, …
  - — criticized in EE, Gegelia, EPJA 41 (09) 341; EE, Gasparyan, Gegelia, Meißner, EPJA 54 (18) 186
  - — in fact, not RG-invariant beyond LO Ashot Gasparyan, EE, to appear

- finite-Λ EFT with $\Lambda \lesssim \Lambda_b \sim 600$ MeV Lepage, EE, Gegelia, Meißner, Reinert, Entem, Machleidt, …
  - — phenomenologically successful; approximate Λ-independence verified a posteriori
  - — renormalizability (in the EFT sense) has been rigorously proven to NLO using the BPHZ subtraction method (forest formula)
    Ashot Gasparyan, EE, PRC 105 (2022) 024001; to appear



Nonperturbative inclusion of pions

LETs for neutron-proton scattering: nonperturbative vs perturbative OPEP

	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
$^1S_0$ partial wave					
LO EE, Gegelia, PLB617 (12) 338	fit	1.50	−1.9	8.6(8)	−37(10)
NLO EE et al., EPJA51 (15) 71	fit	fit	−0.61 ... − 0.55	5.1 ... 5.5	−30.8 ... − 29.6
NLO KSW Cohen, Hansen '98	fit	fit	−3.3	18	−108
Empirical values	−23.7	2.67	−0.5	4.0	−20

	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
$^3S_1$ partial wave					
LO EE, Gegelia, PLB617 (12) 338	fit	1.60	−0.05	0.82	−5.0
NLO Baru et al., PRC94 (16) 014001	fit	fit	0.06	0.70	−4.0
NLO KSW Cohen, Hansen '98	fit	fit	−0.95	4.6	−25
Empirical values	5.42	1.75	0.04	0.67	−4.0

- perturbative inclusion of pions (KSW approach) fails
- $^1S_0$ channel: limited predictive power of the LETs due to the weakness of the OPEP; taking into account the range correction (NLO) leads to improvement
- $^3S_1$ channel: LETs work as advertised (strong tensor part of the OPEP)

Renormalization vs. peratization

EE, Gegelia, EPJA 41 (09) 341

4) How not to renormalize the Schrödinger equation

A toy model with separable interactions: $V(p,\, p') = v_l\, F_l(p)\, F_l(p') + v_s\, F_s(p)\, F_s(p')$

with the form-factors: $F_l(p) \equiv \dfrac{\sqrt{p^2 + m_s^2}}{p^2 + m_l^2}\,, \quad F_s(p) \equiv \dfrac{1}{\sqrt{p^2 + m_s^2}}$

It is convenient to express $v_{l,s}$ in terms of the dimensionless $\alpha_{l,s}$, $a_{l,s} =: \alpha_{l,s}/m_{l,s}$

„Chiral expansion“ of the ERE coefficients:

$$
\begin{aligned}
a &= \frac{1}{m_l}\left(\alpha_a^{(0)} + \alpha_a^{(1)}\frac{m_l}{m_s} + \alpha_a^{(2)}\frac{m_l^2}{m_s^2} + \ldots\right) \\
r &= \frac{1}{m_l}\left(\alpha_r^{(0)} + \alpha_r^{(1)}\frac{m_l}{m_s} + \alpha_r^{(2)}\frac{m_l^2}{m_s^2} + \ldots\right) \\
v_i &= \frac{1}{m_l^{2i-1}}\left(\alpha_{v_i}^{(0)} + \alpha_{v_i}^{(1)}\frac{m_l}{m_s} + \alpha_{v_i}^{(2)}\frac{m_l^2}{m_s^2} + \ldots\right)
\end{aligned}
$$

The dimensionless coefficients $\alpha_a^{(n)}$, $\alpha_r^{(n)}$ and $\alpha_{v_i}^{(n)}$ are determined by the form of the interaction and expressible in terms of $\alpha_{l,s}$.

Renormalization vs. peratization

EE, Gegelia, EPJA 41 (09) 341

For example, the scattering length:

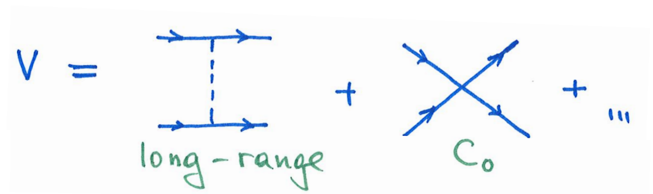
$$\alpha_a^{(0)} = \alpha_l\,, \qquad \alpha_a^{(1)} = (\alpha_l - 1)^2 \alpha_s\,, \qquad \alpha_a^{(2)} = (\alpha_l - 1)^2 \alpha_l \alpha_s^2\,, \qquad \ldots$$

Similarly, for the effective range: $\alpha_r^{(0)} = \dfrac{3\alpha_l - 4}{\alpha_l}\,, \qquad \alpha_r^{(1)} = \dfrac{2\,(\alpha_l - 1)\,(3\alpha_l - 4)\,\alpha_s}{\alpha_l^2}\,,$

$$\alpha_r^{(2)} = \frac{(\alpha_l - 1)\,(3\alpha_l - 4)\,(5\alpha_l - 3)\,\alpha_s^2 + (2 - \alpha_l)\,\alpha_l^2}{\alpha_l^3}\,, \qquad \ldots$$

[Notice: in the considered model, short-range interaction is suppressed. Consequently, the 1st terms in the „chiral expansion“ are determined by the long-range force alone.]

Consider now the effective theory by replacing the short-range interaction by contact terms.



LO: long-range interaction alone, trivially reproduce $\alpha_a^{(0)}$, $\alpha_r^{(0)}$ and $\alpha_{v_i}^{(0)}$ (LETs)

NLO: C_0 is insufficient to absorb all UV divergences $\longrightarrow$ do a finite-Λ theory:

— calculate the amplitude for a fixed Λ,

— renormalize by tuning $C_0(\Lambda)$ to the scattering length (viewed as „datum“)

Renormalization vs. peratization

EE, Gegelia, EPJA 41 (09) 341

According to the LETs, expect to reproduce $\alpha_r^{(1)}$, $\alpha_{v_i}^{(1)}$. E.g. the effective range:

$$r_\Lambda = \frac{1}{m_l}\Big[\underbrace{\frac{3\alpha_l - 4}{\alpha_l}}_{\text{LO LET}} + \underbrace{\frac{2(\alpha_l-1)(3\alpha_l-4)\alpha_s}{\alpha_l^2 m_s}}_{\text{NLO LET}} m_l + \Big(\frac{4(\alpha_l-2)\alpha_s}{\pi\alpha_l m_s^2}\left(\ln\frac{m_s}{2\Lambda}+1\right)$$

$$+ \frac{(\alpha_l-1)(3\alpha_l-4)(5\alpha_l-3)\alpha_s^2 + (2-\alpha_l)\alpha_l^2}{\alpha_l^3 m_s^2}\Big) m_l^2 + \mathcal{O}\left(m_l^3\right)\Big]$$

Works as advertised. Λ-dependence appears in terms beyond the accuracy of the calculation. For $\Lambda \sim m_s$, their contributions are suppressed (NDA).

Infinite-Λ limit (peratization)

Take the limit $T_\infty := \lim_{\Lambda\to\infty} T_\Lambda(p,p)$. Fixing again $C_0(\infty)$ from the scattering length we get Λ-independent predictions for the effective range (and shape parameters):

$$r_\infty = \frac{1}{m_l}\Big[\frac{3\alpha_l - 4}{\alpha_l} + \boxed{\frac{4(\alpha_l-1)^2\alpha_s}{\alpha_l^2 m_s} m_l}$$

$$+ \frac{\alpha_l^3\left(8\alpha_s^2-1\right) + \alpha_l^2\left(2-20\alpha_s^2\right) + 16\alpha_l\alpha_s^2 - 4\alpha_s^2}{\alpha_l^3 m_s^2} m_l^2 + \ldots\Big]$$

← **while Λ-independent, the results violate the LETs which is unacceptable from the EFT point of view**

Summary of part I

- Long-range interactions control the near-threshold energy behavior of the amplitude and lead to LETs. Application of the LETs to NN scattering suggests that the OPEP has to be treated non-perturbatively in the $^3S_1$-$^3D_1$ channel.

- Iterations of the OPEP in the LS equation require an infinite number of counter terms. It is not known how to subtract all UV divergences in that case. Thus, one has to work with a finite cutoff Λ of the order of the breakdown scale Λ_b (implicit renormalization by tuning bare LECs to experimental data).

- Choosing the cutoff $\Lambda \gg \Lambda_b$ without including ALL counter terms necessary to absorb the UV divergences is generally dangerous (even is the $\Lambda \to \infty$ limit for the amplitude exists…).

See also:
EE, Gasparyan, Gegelia, Meißner, „How (not) to renormalize integral equations with singular potentials in EFT“, Eur. Phys. J. A54 (2018) 186.

Part II: Methods

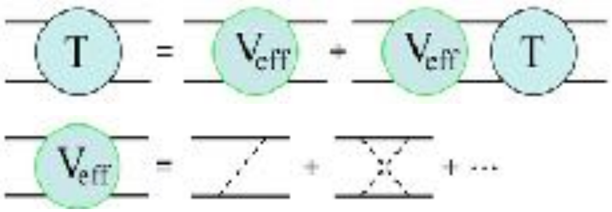
How to derive nuclear forces and currents?

Introduction

1. Introduction

Recall our framework:

$$\left[\left(\sum_{i=1}^{A}\frac{-\vec{\nabla}_i^2}{2m_N}+\mathcal{O}(m_N^{-3})\right)+\underbrace{V_{2N}+V_{3N}+V_{4N}+\ldots}_{\text{derived in ChPT}}\right]|\Psi\rangle = E|\Psi\rangle$$



Nuclear forces & currents are defined as irreducible contributions to the amplitude [i.e. the ones which are not generated by iterations of the LS equation].

They can be derived using a variety of methods including [In all cases, utilize a perturbative expansion within ChPT]:

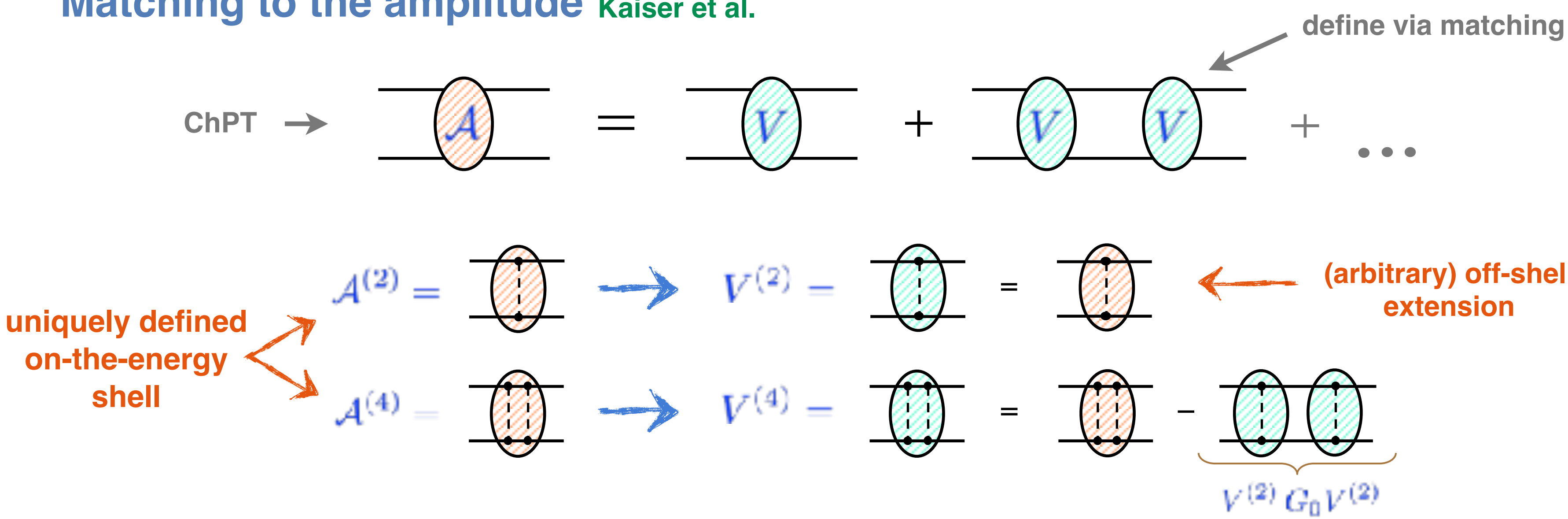
— S-matrix matching **[Kaiser et al.]**
— time-ordered perturbation theory **[Pastore, Baroni, Schiavilla et al.]**
— method of unitary transformations (UTs) **[EE, Glöckle, Meißner, Krebs, Kölling]**

More challenging than just calculating Feynman diagrams:

— need to subtract reducible pieces in order to avoid double counting
— have to deal with non-uniqueness of nuclear potentials
— want to maintain renormalizability

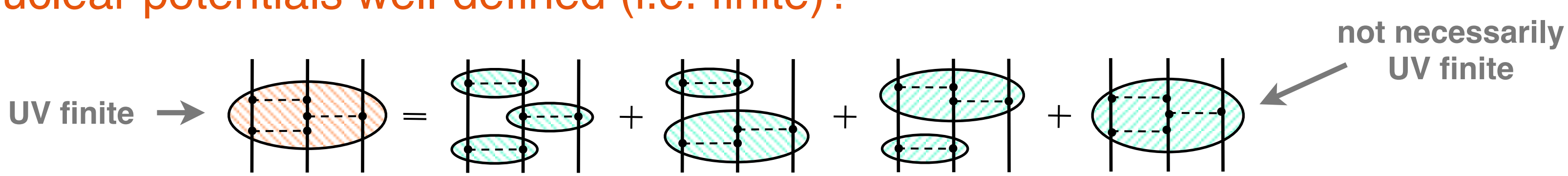
Introduction

Matching to the amplitude Kaiser et al.



Higher-order terms in the Hamiltonian „know" about the choice made for the off-shell extension (consistency...)

Are nuclear potentials well-defined (i.e. finite)?

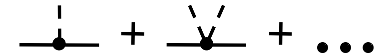


So far, it was always possible to renormalize nuclear forces by systematically exploiting their unitary ambiguity…

Method of UT

2. Method of unitary transformation

Taketani, Mashida, Ohnuma'52; Okubo '54; EE, Glöckle, Meißner, Krebs, Kölling, ...

(i) Canonical transformation & quantization: $\mathcal{L}_{\pi N} \longrightarrow \mathcal{H}_{\pi N} =$ 

nucleonic states $|N\rangle, |NN\rangle, \ldots$

EOM: $$\begin{pmatrix} \eta H \eta & \eta H \lambda \\ \lambda H \eta & \lambda H \lambda \end{pmatrix} \begin{pmatrix} |\phi\rangle \\ |\psi\rangle \end{pmatrix} = E \begin{pmatrix} |\phi\rangle \\ |\psi\rangle \end{pmatrix}$$ ⟵ can not solve (infinite-dimensional eq.)

projectors *states with mesons* $|N\pi\rangle, |N\pi\pi\rangle, \ldots$

(ii) Decouple pions via a suitable UT: $$\tilde{H} \equiv U^\dagger \begin{pmatrix} \eta H \eta & \eta H \lambda \\ \lambda H \eta & \lambda H \lambda \end{pmatrix} U = \begin{pmatrix} \eta \tilde{H} \eta & 0 \\ 0 & \lambda \tilde{H} \lambda \end{pmatrix}$$

Minimal parametrization: $$U = \begin{pmatrix} \eta(1 + A^\dagger A)^{-1/2} & -A^\dagger (1 + A A^\dagger)^{-1/2} \\ A(1 + A^\dagger A)^{-1/2} & \lambda (1 + A A^\dagger)^{-1/2} \end{pmatrix}, \quad A = \lambda A \eta$$

Okubo '54

Require: $\eta \tilde{H} \lambda = \lambda \tilde{H} \eta = 0$ ⟶ $$\lambda \left(H - [A, H] - A H A \right) \eta = 0$$

The major problem is to solve the nonlinear decoupling equation.

Notice: similar methods widely used in nuclear & many-body physics (Lee-Suzuki)

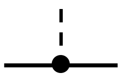
Method of UT

Once the operator A is calculated, nuclear forces are determined via:

$$V = \eta(\tilde{H} - H_0) = \eta\Big[(1 + A^\dagger A)^{-1/2}(H + A^\dagger H + HA + A^\dagger H A)(1 + A^\dagger A)^{-1/2} - H_0\Big]\eta$$

In QFT, block-diagonalization of the Hamilton operator can usually only be achieved in perturbation theory…

Example: expansion in powers of the coupling constant

$H_I =$  $\propto g \longrightarrow$ ansatz: $A = A^{(1)} + A^{(2)} + A^{(3)} + \ldots$

Recursive solution of the decoupling equation $\lambda\,(H - [A,\, H] - AHA)\,\eta = 0$

$$g^1 : \quad \lambda(H_I - [A^{(1)},\, H_0])\eta = 0 \quad \longrightarrow \quad A^{(1)} = \lambda \frac{H_I}{E_\eta - E_\lambda}\eta$$

$$g^2 : \quad \lambda(H_I A^{(1)} - [A^{(2)},\, H_0])\eta = 0 \quad \longrightarrow \quad A^{(2)} = \lambda \frac{H_I A^{(1)}}{E_\eta - E_\lambda}\eta$$

$\ldots$

Method of UT

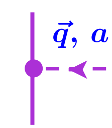
One then obtains:

$$V_{\rm eff} = \eta\Big[(1+A^\dagger A)^{-1/2}(H + A^\dagger H + HA + A^\dagger H A)(1+A^\dagger A)^{-1/2} - H_0\Big]\eta$$
$$= \eta A^{(1)^\dagger}\lambda H_I \eta \;+\; \eta H_I \lambda A^{(1)}\eta \;+\; \eta A^{(1)^\dagger}\lambda H_0 A^{(1)}\eta \;+\; \mathcal{O}(g^3)$$

In the static approximation (i.e. in the limit $m \to \infty$), one has:

$$E_\eta - E_\lambda \simeq E_\pi \equiv \omega \quad \longrightarrow \quad V_{\rm eff}^{(2)} = -\eta H_I \frac{\lambda}{\omega} H_I \eta$$

Take the LO πN vertex: $\mathcal{H}_\mathcal{I} = -\mathcal{L}_\mathcal{I} = \dfrac{g_A}{2F_\pi} N^\dagger \left(\vec\sigma\vec\tau \cdot \cdot \vec\nabla \vec\pi\right) N$

$$\langle \vec p\,' | H_I | \vec p;\ \vec q, a\rangle \;=\; \boxed{\frac{i g_A}{2F_\pi}\frac{1}{\sqrt{2\omega_q}}\,\vec\sigma\cdot\vec q\,\tau^a} \times \delta^3(\ldots) \;\longrightarrow$$ Feynman-like rule for 

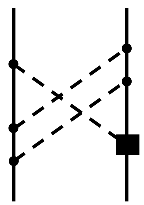
— The chiral expansion can be cast into the form similar to the expansion in powers of the coupling constant. **EE, Eur. Phys. J. A 34 (2007) 197**

— The resulting potentials at N^{3}LO and beyond calculated using DR can ***not*** be made finite — enforce renormalizability by exploiting unitary ambiguity.

Merging MUT with ChPT

3. Merging MUT with ChPT [EE, Glöckle Meißner '98; EE, EPJA 34 (2007) 197]

Chiral expansion is **not** an expansion in powers of momenta. How to derive nuclear forces utilizing the chiral expansion?



$\sim \left(\frac{Q}{\Lambda}\right)^{\nu}$ where $\nu = 2 - N + 2L + \sum_i V_i \Delta_i$ and $\Delta_i = -2 + \frac{1}{2} n_i + d_i$

# of loops (L); *# of vertices of type Δ_i* (V_i); *# of nucleon field operators* (n_i); *# of derivatives* (d_i)

Perfect for diagrams, but inconvenient for solving $\lambda(H - [A, H] - AHA)\eta = 0$

Rewrite in a more convenient way. Trick: count powers of the *hard* scale Λ_b rather than of the soft scale Q. The only way for Λ_b to emerge is through the LECs of the effective Lagrangian. Thus, the power ν is given by:

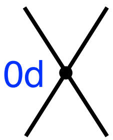
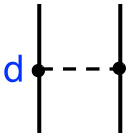
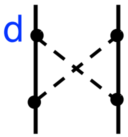
$$\nu = -2 + \sum_i V_i \kappa_i$$

where κ is an inverse mass dimension of the coupling constant of a vertex i, $[c_i] = (mass)^{-\kappa_i}$

$$\mathcal{L}_i = c_i \, (N^\dagger(\ldots)N)^{\frac{n_i}{2}} \, \pi^{p_i} \, (\partial_\mu, M_\pi)^{d_i} \quad \longrightarrow \quad \kappa_i = d_i + \frac{3}{2} n_i + p_i - 4$$

Merging MUT with ChPT

Examples:

	0d  $\sim Q^0$	1d  $\sim Q^0$ ν = 2 [derivatives] – 2 [π-propagator]	1d  $\sim Q^2$ ν = 4 [loop integral] + 4 [derivatives] – 4 [2 π-propagators] – 2 [2 HB nucl. prop.]
$\Delta_i = -2 + \frac{1}{2} n_i + d_i$ $\nu = 2 - N + 2L + \sum_i V_i \Delta_i$	Δ = -2 + 2 + 0 = 0 ν = 2 - 2 + 0 + 0 = 0	Δ = -2 + 1 + 1 = 0 ν = 2 - 2 + 0 + 2*0 = 0	Δ = -2 + 1 + 1 = 0 ν = 2 - 2 + 2 + 4*0 = 2
$\kappa_i = d_i + \frac{3}{2} n_i + p_i - 4$ $\nu = -2 + \sum_i V_i \kappa_i$	κ = 0 + 6 + 0 - 4 = 2 ν = -2 + 2 = 0	κ = 1 + 3 + 1 - 4 = 1 ν = -2 + 2*1 = 0	κ = 1 + 3 + 1 - 4 = 1 ν = -2 + 4*1 = 2

Chiral symmetry ensures that only non-renormalizable interactions with $\kappa > 0$ (i.e. the irrelevant interactions) appear in $\mathcal{L}_{\rm eff}$ ⟶ perturbative expansion for nuclear forces

Merging MUT with ChPT

\# of derivatives and/or M_π-insertions — terms with more pion fields

$$\mathcal{L}^{(1)}_{\pi N} = N^\dagger \Big[\underbrace{i\partial_0 - \frac{g_A}{2F}\boldsymbol{\tau}\vec{\sigma}\cdot\vec{\nabla}\boldsymbol{\pi}}_{\kappa=1} - \underbrace{\frac{1}{4F^2}\boldsymbol{\tau}\times\boldsymbol{\pi}\cdot\dot{\boldsymbol{\pi}}}_{\kappa=2} + \underbrace{\frac{g_A}{4F^3}\Big((4\alpha-1)\boldsymbol{\tau}\cdot\boldsymbol{\pi}(\boldsymbol{\pi}\vec{\sigma}\cdot\vec{\nabla}\boldsymbol{\pi}) + 2\alpha\pi^2(\boldsymbol{\tau}\vec{\sigma}\cdot\vec{\nabla}\boldsymbol{\pi})\Big)}_{\kappa=3} + \ldots \Big] N$$

$$\mathcal{L}^{(2)}_{\pi N} = N^\dagger \Big[\underbrace{4M^2 c_1}_{\kappa=1} - \underbrace{\frac{2c_1}{F^2}M^2\pi^2 + \frac{c_2}{F^2}\dot{\pi}^2 + \frac{c_3}{F^2}(\partial_\mu\boldsymbol{\pi})\cdot(\partial^\mu\boldsymbol{\pi}) - \frac{c_4}{4F^2}(\boldsymbol{\tau}\vec{\sigma}\times\vec{\nabla}\boldsymbol{\pi})\cdot\vec{\nabla}\boldsymbol{\pi}}_{\kappa=3} + \ldots \Big] N$$

$$\mathcal{L}^{(0)}_{NN} = \underbrace{\frac{1}{2}C_S N^\dagger N\, N^\dagger N + \frac{1}{2}C_S N^\dagger\vec{\sigma}N\cdot N^\dagger\vec{\sigma}N}_{\kappa=2}$$

...

Expansion in the coupling constant ⟷ expansion in the inverse mass dimension

$$H_I = \sum_{\kappa=1}^{\infty} H^{(\kappa)} \longrightarrow$$ a more general ansatz: $$A = \sum_{\alpha=1}^{\infty} A^{(\alpha)}$$

Recursive solution of the decoupling equation:

$$A^{(\alpha)} = -\frac{1}{E_\lambda}\lambda\Big[H^{(\alpha)} + \sum_{i=1}^{\alpha-1}H^{(i)}A^{(\alpha-i)} - \sum_{i=1}^{\alpha-1}A^{(\alpha-i)}H^{(i)} - \sum_{i=1}^{\alpha-2}\sum_{j=1}^{\alpha-j-1}A^{(i)}H^{(j)}A^{(\alpha-i-j)}\Big]\eta$$

$$\longrightarrow \tilde{V}^{\mathrm{UT}}_{\mathrm{eff}} = \ldots$$

Easy to implement in FORM, MATHEMATICA, ...

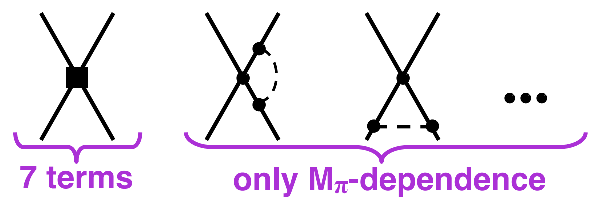
NLO corrections to the nuclear force

4. Example: NLO corrections to the nuclear force

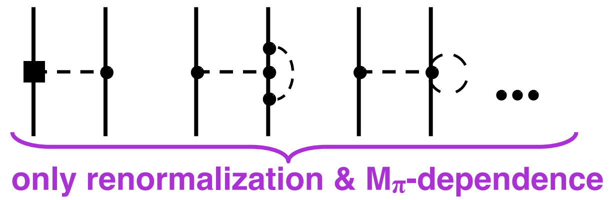
At LO (v = 0) one has: $V_{\rm NN}^{(0)} = -\left(\frac{g_A}{2F_\pi}\right)^2 \frac{\vec{\sigma}_1 \cdot \vec{q}\, \vec{\sigma}_2 \cdot \vec{q}}{q^2 + M_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 + C_S + C_T \vec{\sigma}_1 \cdot \vec{\sigma}_2$

The first corrections come at order v = 2:

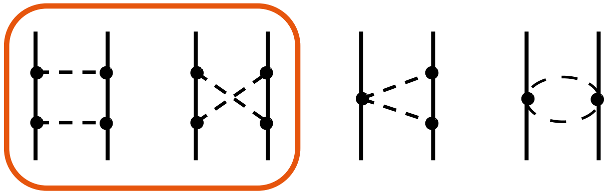
- 1-loop corrections to the LO contacts



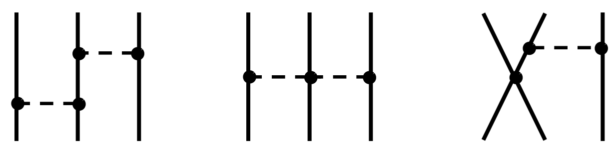
- 1-loop corrections to the OPEP
  [renormalization within the MUT is described in: EE, Glöckle, Meißner, NPA714 (2003) 535]



- Leading two-pion exchange potential



- Contributions from 3N diagrams cancel completely at this order (no 3NF@NLO)



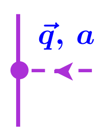
NLO corrections to the nuclear force

As an example, let us work out the TPEP $\propto g_A^4$ using the MUT. Extending the perturbative calculations to 4th order, one finds the relevant operators:

$$V^{(2)} = \eta \Big[- H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)} + \underbrace{\frac{1}{2} H_I^{(1)} \frac{\lambda}{E_\pi^2} H_I^{(1)} \, \eta \, H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)} + \frac{1}{2} H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)} \, \eta \, H_I^{(1)} \frac{\lambda}{E_\pi^2} H_I^{(1)}}_{\text{ensure the unitarity of the transformation („wave-function orthonormalization“)}} \Big] \eta$$

These Fock-space operators give rise to 1N, 2N and 3N operators. Here, we focus only on the 2N contributions from 2π-exchange.

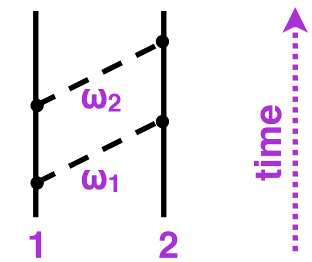
In principle, we have to express $H_I^{(1)}$ in terms of creation/destruction operators and evaluate matrix elements $\langle \vec{p}_1{}' \, \vec{p}_2{}' \, | V^{(2)} | \vec{p}_1 \, \vec{p}_2 \rangle$. It is, however, more efficient to use the already introduced Feynman-like rule:

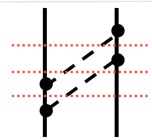
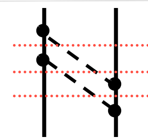
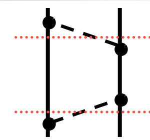
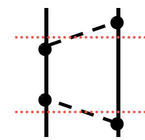
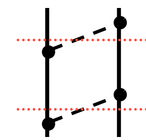
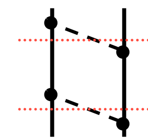


$$\longrightarrow \quad \frac{i g_A}{2 F_\pi} \frac{1}{\sqrt{2\omega_q}} \, \vec{\sigma} \cdot \vec{q} \, \tau^a$$

NLO corrections to the nuclear force

Consider time-ordered graphs of the planar-box type and assign the label „1“ to the pion which is emitted by nucleon 1 first. Since all such diagrams have the same spin-isospin-momentum structure, we can (first) collect and simplify the energy denominators:

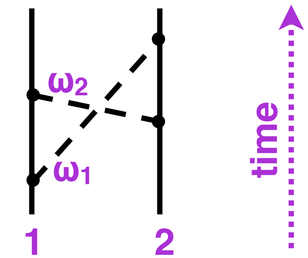


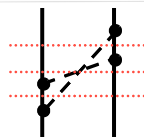
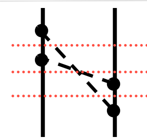
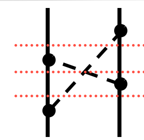
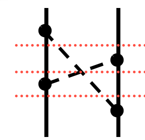
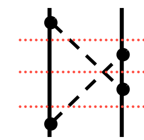
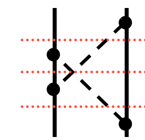
operators \ graphs						
$-H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}$	$-\frac{1}{\omega_1\omega_{12}\omega_2}$	$-\frac{1}{\omega_1\omega_{12}\omega_2}$	—	—	—	—
$\frac{1}{2}H_I^{(1)}\frac{\lambda}{E_\pi^2}H_I^{(1)}\,\eta\,H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}$	—	—	$\frac{1}{2\omega_1\omega_2^2}$	$\frac{1}{2\omega_1\omega_2^2}$	$\frac{1}{2\omega_1\omega_2^2}$	$\frac{1}{2\omega_1\omega_2^2}$
$\frac{1}{2}H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}\,\eta\,H_I^{(1)}\frac{\lambda}{E_\pi^2}H_I^{(1)}$	—	—	$\frac{1}{2\omega_1^2\omega_2}$	$\frac{1}{2\omega_1^2\omega_2}$	$\frac{1}{2\omega_1^2\omega_2}$	$\frac{1}{2\omega_1^2\omega_2}$

Here, $\omega_{12} := \omega_1 + \omega_2$. Collecting all denominators, we obtain: $2\frac{\omega_1^2+\omega_1\omega_2+\omega_2^2}{\omega_1^2\omega_2^2(\omega_1+\omega_2)}$

NLO corrections to the nuclear force

Use the same bookkeeping for the crossed-box diagram and proceed in the same way to collect energy denominators.

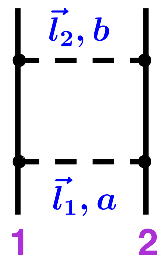


graphs / operators						
$-H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}$	$-\frac{1}{\omega_1^2\omega_{12}}$	$-\frac{1}{\omega_{12}\omega_2^2}$	$-\frac{1}{\omega_1^2\omega_{12}}$	$-\frac{1}{\omega_{12}\omega_2^2}$	$-\frac{1}{\omega_1\omega_{12}\omega_2}$	$-\frac{1}{\omega_1\omega_{12}\omega_2}$
$\frac{1}{2}H_I^{(1)}\frac{\lambda}{E_\pi^2}H_I^{(1)}\,\eta\,H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}$	—	—	—	—	—	—
$\frac{1}{2}H_I^{(1)}\frac{\lambda}{E_\pi}H_I^{(1)}\,\eta\,H_I^{(1)}\frac{\lambda}{E_\pi^2}H_I^{(1)}$	—	—	—	—	—	—

Collecting all denominators, we obtain: $$-2\frac{\omega_1^2+\omega_1\omega_2+\omega_2^2}{\omega_1^2\omega_2^2(\omega_1+\omega_2)}$$

NLO corrections to the nuclear force

Let's put everything together:

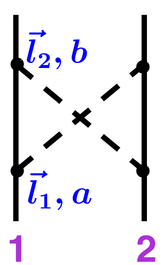


convention — collected energy denominators

$$\underbrace{(2\pi)^3}_{\text{convention}} \int \frac{d^3 l_1}{(2\pi)^3} \frac{d^3 l_2}{(2\pi)^3} \, \delta^3(\vec{q} - \vec{l}_1 - \vec{l}_2) \; \underbrace{2\frac{\omega_1^2 + \omega_1\omega_2 + \omega_2^2}{\omega_1^2\omega_2^2(\omega_1 + \omega_2)}}_{\text{collected energy denominators}}$$

$$\times \underbrace{\left(\frac{i g_A}{2F_\pi}\right)^4 \frac{1}{4\omega_1\omega_2} (\vec{\sigma}_1 \cdot \vec{l}_1 \tau_1^a)\, (\vec{\sigma}_1 \cdot \vec{l}_2 \tau_1^b)\, (-\vec{\sigma}_2 \cdot \vec{l}_1 \tau_2^a)\, (-\vec{\sigma}_2 \cdot \vec{l}_2 \tau_2^b)}_{\text{vertex factors from Feynman-like rules}}$$

vertex factors from Feynman-like rules



$$(2\pi)^3 \int \frac{d^3 l_1}{(2\pi)^3} \frac{d^3 l_2}{(2\pi)^3} \, \delta^3(\vec{q} - \vec{l}_1 - \vec{l}_2) \; (-2)\frac{\omega_1^2 + \omega_1\omega_2 + \omega_2^2}{\omega_1^2\omega_2^2(\omega_1 + \omega_2)}$$

$$\times \left(\frac{i g_A}{2F_\pi}\right)^4 \frac{1}{4\omega_1\omega_2} (\vec{\sigma}_1 \cdot \vec{l}_1 \tau_1^a)\, (\vec{\sigma}_1 \cdot \vec{l}_2 \tau_1^b)\, (-\vec{\sigma}_2 \cdot \vec{l}_2 \tau_2^b)\, (-\vec{\sigma}_2 \cdot \vec{l}_1 \tau_2^a)$$

where $\vec{q} := \vec{p}_1{}' - \vec{p}_1$ is the momentum transfer.

NLO corrections to the nuclear force

Performing spin-isospin algebra, switching to new momenta $\vec{q}\,' = \vec{l}_1 + \vec{l}_2$ and $\vec{l} = \vec{l}_1 - \vec{l}_2$ [don't forget the Jacobi-determinant…] and performing the trivial integration over $\vec{q}\,'$, one obtains:

$$V = -\left(\frac{g_A}{2F_\pi}\right)^4 \int \frac{d^3 l}{(2\pi)^3} \frac{\omega_+^2 + \omega_+\omega_- + \omega_-^2}{\omega_+^3 \omega_-^3 (\omega_+ + \omega_-)} \left[\frac{\vec{\tau}_1 \cdot \vec{\tau}_2}{2} \left(l^2 - q^2\right)^2 + 3 \vec{\sigma}_1 \cdot \vec{q} \times \vec{l}\, \vec{\sigma}_2 \cdot \vec{q} \times \vec{l}\right]$$

where I have introduced $\omega_\pm := \sqrt{(\vec{q} \pm \vec{l}\,)^2 + 4 M_\pi^2}$.

Tricks to evaluate the loop integral [for more tricks see e.g.: Rijken, Ann. Phys. 208 (1991) 253]

- Express the energy factor as a product of pion propagators. Use:

$$\frac{\omega_+^2 + \omega_+\omega_- + \omega_-^2}{\omega_+^3 \omega_-^3 (\omega_+ + \omega_-)} = \frac{1}{4M_\pi} \frac{\partial}{\partial M_\pi} \frac{1}{\omega_+\omega_-(\omega_+ + \omega_-)}$$

$$\frac{1}{\omega_+\omega_-(\omega_+ + \omega_-)} = \frac{2}{\pi} \int_0^\infty \frac{d\lambda}{(\omega_+^2 + \lambda^2)(\omega_-^2 + \lambda^2)}$$

- Combine the propagators by introducing the Feynman parameter and do DR

NLO corrections to the nuclear force

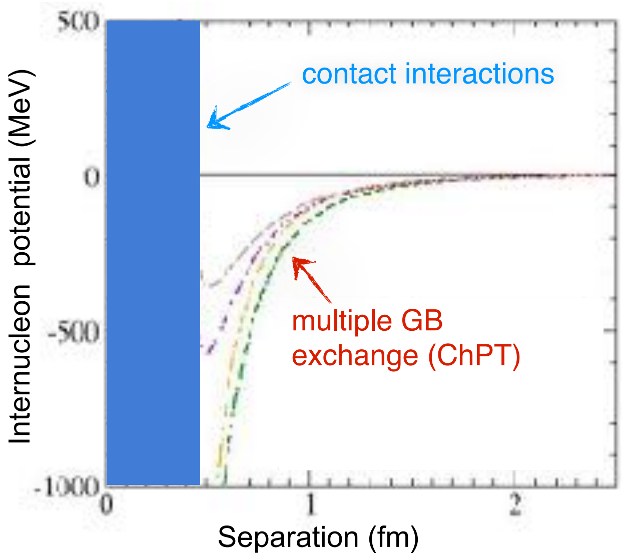
The final result is:

$$V = -\frac{g_A^4}{384\pi^2 F_\pi^4}\Big[\vec{\tau}_1\cdot\vec{\tau}_2\Big(20M_\pi^2 + 23q^2 + \frac{48M_\pi^4}{4M_\pi^2+q^2}\Big) + 18\big(\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q} - q^2\vec{\sigma}_1\cdot\vec{\sigma}_2\big)\Big]L(q) + \underbrace{\ldots}_{\text{contact terms with up to 2 momenta}}$$

where the loop function is given by: $L(q) = \frac{1}{q}\sqrt{4M_\pi^2+q^2}\,\ln\frac{\sqrt{4M_\pi^2+q^2}+q}{2M_\pi}$

Notice:

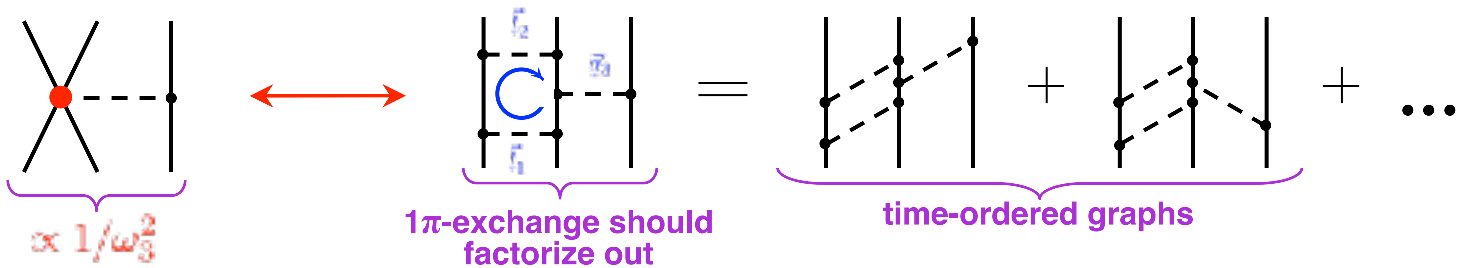
- van der Waals-like forces in r-space. At large distances $\sim \exp(-2M_\pi r)$. Highly singular at short distances, $\sim r^{-3-\nu}$ (ν is the chiral order).
- χ-expansion meaningful (convergent) for $r \gtrsim M_\pi^{-1}$.



A note on renormalization

5. A note on renormalization

All UV divergences in the nuclear potentials up to N²LO are removed by the corresponding counterterms. This is not necessarily true starting from N³LO…



$$V = \ldots = \int d^3 l_1 \, d^3 l_2 \, \delta(\vec{l}_1 - \vec{l}_2 - \vec{q}_1) \left[\ldots \right]$$

$$\times \left[2\frac{\omega_1^2 + \omega_2^2}{\omega_1^4 \, \omega_2^4 \, \omega_3^2} + \frac{8}{\omega_1^2 \, \omega_2^2 \, \omega_3^4} - \frac{\omega_1 + \omega_2}{\omega_1^3 \, \omega_2^3 \, \omega_3^3} - \frac{2}{\omega_1^4 \, \omega_2^2 \, \omega_3 \, (\omega_1 + \omega_3)} - \frac{2}{\omega_1^2 \, \omega_2^4 \, \omega_3 \, (\omega_2 + \omega_3)} \right]$$

$\sqrt{\vec{l}_{1,2}^{\,2} + M_\pi^2}$

→ cannot renormalize the potential !

A note on renormalization

Solution [EE '06]

Nuclear potentials are not uniquely defined. Starting from N³LO, can construct additional UTs in Fock space beyond the (minimal) Okubo UT.

The UTs relevant for the N³LO contributions $\propto g_A^6$ are $U = e^{\alpha_1 S_1 + \alpha_2 S_2}$, with the generators given by:

$$S_1 = \eta\Big[H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)}\, \eta\, H_I^{(1)} \frac{\lambda}{E_\pi^3} H_I^{(1)} - \mathrm{h.\,c.}\Big]\eta$$

$$S_2 = \eta\Big[H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)} \frac{\lambda}{E_\pi^2} H_I^{(1)} - \mathrm{h.\,c.}\Big]\eta$$

They induce additional contributions in the Hamiltonian starting from N³LO

$$\delta V^{(4)} = [(H_{\mathrm{kin}} + V^{(0)}),\ S] = -\alpha_1\, H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)}\, \eta\, H_I^{(1)} \frac{\lambda}{E_\pi} H_I^{(1)}\, \eta\, H_I^{(1)} \frac{\lambda}{E_\pi^3} H_I^{(1)} + \ldots .$$

Demanding renormalizability constrains α_1, α_2 and leads to unique static results.

So far, it was always possible to get finite nuclear potentials & currents.

6. Electroweak currents

- Switch on external sources $s,\ p,\ r_\mu,\ l_\mu$ and consider ***local*** chiral rotations:

$$r_\mu \rightarrow r'_\mu = R\, r_\mu R^\dagger + iR\,\partial_\mu R^\dagger\,, \qquad l_\mu \rightarrow l'_\mu = L\, l_\mu L^\dagger + iL\,\partial_\mu L^\dagger\,,$$
$$s + i\,p \rightarrow s' + i\,p' = R(s+i\,p)L^\dagger\,, \qquad s - i\,p \rightarrow s' - i\,p' = L(s-i\,p)R^\dagger$$

The sources can be conveniently rewritten via $v_\mu = \frac{1}{2}(r_\mu + l_\mu)\,, \quad a_\mu = \frac{1}{2}(r_\mu - l_\mu)$ with:

$$v_\mu = v_\mu^{(s)} + \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{v}_\mu, \qquad a_\mu = \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{a}_\mu, \qquad s = s_0 + \boldsymbol{\tau}\cdot\boldsymbol{s}, \qquad p = p_0 + \boldsymbol{\tau}\cdot\boldsymbol{p}$$

already known from the strong sector…

- (Naive) attempt: calculate $\tilde{H} \rightarrow \tilde{H}[a,v,s,p] = U^\dagger H[a,v,s,p]U$ and extract the nuclear currents via $V_\mu^a(\vec{x}\,) = \dfrac{\delta\tilde{H}}{\delta v_a^\mu(\vec{x},t)}, \quad A_\mu^a(\vec{x}\,) = \dfrac{\delta\tilde{H}}{\delta a_a^\mu(\vec{x},t)}$ at $v = a = p = \boldsymbol{s} = 0, \quad s_0 = m_q.$

However, the resulting currents turn out to be non-renormalizable…

⟶ Need to consider a more general class of UTs

Specifically, employ additional η-space UTs $U[a,v,s,p]$ subject to the constraint $U[0,0,m_q,0]=1$

Notice: the resulting UTs are time-dependent, thus $H' \neq U^\dagger H U$. Indeed:

$$i\frac{\partial}{\partial t}\Psi = H\Psi \quad \longrightarrow \quad i\frac{\partial}{\partial t}\left(U^\dagger(t)\Psi\right) = \left[U^\dagger(t)HU(t) - U^\dagger(t)\left(i\frac{\partial}{\partial t}U(t)\right)\right]\left(U^\dagger(t)\Psi\right)$$

- Continuity equations = manifestations of the χ symmetry Krebs, EE, Meißner, Annals Phys. 378 (17) 317

Summary of part II

- Nuclear forces and currents are not unique (off-shell behavior). It is crucial to maintain consistency.
- MUT can be combined with the chiral expansion and provides a convenient approach to derive nuclear forces and currents.
- Renormalizability of the nuclear potentials is not automatically guaranteed starting from N^3LO but can be maintained by systematically exploiting the unitary ambiguity.